MULTI-VALUED GRAPHS IN EMBEDDED CONSTANT MEAN CURVATURE DISKS

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Abstract. In this paper we prove that an embedded constant mean curvature disk with Gaussian curvature large at a point contains a multi-valued graph around that point on the scale of $|A|^2$. This generalizes Colding and Minicozzi's result for minimal surfaces.

Introduction

In this paper we prove that an embedded and simply connected constant mean curvature (CMC) surface with Gaussian curvature large at a point contains a multi-valued graph around that point on the scale of $|A|^2$, where $|A|^2$ is the norm squared of the second fundamental form. More precisely, our main result is the following:

Theorem 0.1. *Given $N \in \mathbb{Z}_+$, $\omega > 1$ and $\varepsilon > 0$, there exist $C = C(N, \omega, \varepsilon) > 0$, $H > 0$ and $\bar{l} > 1$ so:*
Let $\Sigma \subset \mathbb{R}^3$ be an embedded and simply connected constant mean curvature equal to h surface. If $|h| < \frac{H}{r_0}$ and

$$\sup_{\Sigma \cap B_{r_0 \bar{l}}(0)} |A|^2 \leq 4C^2 r_0^{-2} \text{ and } |A|^2(0) = C^2 r_0^{-2}$$

for some $r_0 > 0$, then Σ (after a rotation) contains an N-valued graph $\Sigma_g \subset \Sigma$ over $D_{\omega \bar{R}} \backslash D_{\bar{R}}$ where $\bar{R} < \frac{r_0}{\omega}$ (with gradient $\leq \varepsilon$ and $\text{dist}_\Sigma(0, \Sigma_g) \leq 4\bar{R}$).

Roughly speaking, to contain a multi-valued graph (Def. 2.2 in this paper) means that locally the surface spirals like a helicoid, Fig. 1. The helicoid is a minimal surface

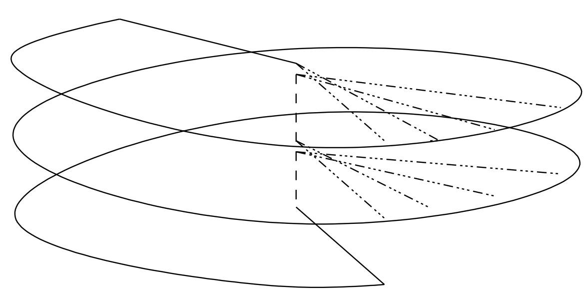

Figure 1. Half of the the Helicoids

parameterized in the following way

$$(s \sin t, s \cos t, t) \qquad \text{where } (s, t) \in \mathbb{R}^2.$$

Our result is a generalization of Colding and Minicozzi's result [4, **Theorem 0.4.**] (Thm. 2.1 in this paper) which is a key ingredient in their series of papers [3, 4, 5, 6] that dealt with the structure of embedded minimal disks. We prove that under equivalent local conditions an embedded CMC disk contains a multi-valued graph as well. For a minimal surface, Colding and Minicozzi were able to extend the multi-valued graph that forms locally, all the way up to the boundary [3]. It is not known if the same can be done for CMC surfaces. This result can be thought as a first step towards a classification of singularities for sequences of embedded CMC disks; indeed, much more needs to be done in this direction.

The proof is by contradiction using a new compactness argument that does not require a bound on the area. The idea is the following: Assuming that Theorem 0.1 is false, we build a sequence Σ_n of embedded CMC disks where each disk satisfies the hypotheses of the theorem with C fixed large and $H = \frac{1}{n}$ but does not contain a N-valued graph. We prove that Σ_n converges C^2 to a minimal surface Σ_∞ which contains an N-valued graph.

Definition 0.2. *A sequence Σ_n of surfaces converges to a surface Σ_∞ in the C^k topology if at any point $p \in \Sigma_\infty$ each Σ_n is locally (near p) a graph over the tangent space $T_p\Sigma_\infty$ and the graph of Σ_n converges to the graph of Σ_∞ in the usual C^k topology.*

We will also consider sequences that converge with multiplicity. This means that we allow more than one graph in the previous definition.

Essentially we show that Σ_n comes as close as we want to its limit and that the limit is an embedded minimal disk which contains an N-valued graph because of **Theorem 0.4.** in [4], therefore so do the CMC surfaces. To create the N-valued graph in the CMC sequence we basically push the multi-valued graph from the minimal surface onto Σ_n. This contradiction proves the theorem.

The difficult part of the proof is to show that the limit is both an embedded surface and simply connected and not, for instance, a minimal lamination or a minimal surface which is not simply connected. Some sort of C^2 convergence follows in a standard way from the bound on the curvature and trivially, since we are assuming that the mean curvature goes to zero, the limit is a minimal object. To assure that the limit is embedded and simply connected we need a uniform upper bound on the number of graphs in Def. 0.2. We investigate the strong stability for a constant mean curvature surface to find out when a CMC surface, which is already a critical point for a certain area functional, is an actual minimum.

First, we prove that, under certain conditions, if two CMC surfaces are close and disjoint, they are almost-stable.

Second, we rule out the possibility that Σ_n contains a large, almost-stable domain ("almost a minimum"), for n large.

Third, we show that if there is not a uniform upper bound on the number of pieces, then a large piece of Σ_n is a graph over another piece, creating a large almost-stable domain and giving the contradiction.

Once the uniform upper bound on the number of pieces is obtained, the convergence to an embedded minimal surface follows. We have to use some topological results to prove that the limit minimal surface is simply connected.

In the first section we provide a short overview of constant mean curvature surfaces. In the second section we describe what a multi-valued graph is and go over the hypotheses of the main result. We also take a closer look at the proof. In the third section we deal with the δ-stability for CMC surfaces, and give a criteria to find δ-stable domains in CMC surfaces. In the fourth section we show how, because of the upper-bound on $|A|^2$, our CMC disk is "uniformly locally flat" and we give a criteria to find large δ-stable domains. In the fifth section we prove that a large δ-stable domain cannot be contained in Σ_n for n large and how this gives an upper bound on the number of graphs. In the sixth section we prove that the limit is an embedded minimal disk and from that we build a multi-valued graph in Σ_n.

We actually prove the result when r_0 in Theorem 0.1 is fixed and equal to one. The main result will follow by rescaling and in Appendix B we describe the rescaling argument for constant mean curvature graphs. In Appendix A we provide examples of CMC surfaces containing arbitrary large multi-valued graphs.

1. **Constant Mean Curvature Surfaces.**

This section is a short review of general properties of CMC surfaces.

Let $\Sigma \subset \mathbb{R}^3$ be a 2-dimensional smooth orientable surface (possibly with boundary) with unit normal N_Σ. Given a function ϕ in the space $C_0^\infty(\Sigma)$ of infinitely differentiable (i.e., smooth), compactly supported functions on Σ, consider the one-parameter variation

$$\Sigma_{t,\phi} = \{x + t\phi(x)N_\Sigma(x) | x \in \Sigma\}$$

and let $A(t)$ be the area functional,

$$A(t) = Area(\Sigma_{t,\phi}).$$

The so-called first variation formula of area is the equation (integration is with respect to $d$area)

$$A'(0) = \int_\Sigma \phi H, \tag{1.1}$$

where H is the mean curvature of Σ. When H is constant the surface is said to be a *constant mean curvature* (CMC) surface [14] and it is a critical point for the area functional restricted to those variations which preserve the *enclosed volume*, in other words ϕ must satisfy the condition,

$$\int_\Sigma \phi = 0.$$

In general, if Σ is given as graph of a function u then

$$H = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right). \tag{1.2}$$

Therefore, when H is constant u satisfies a quasi-linear differential equation. In the particular case where the mean curvature H is identically zero the surface Σ is said to be a *minimal* surface [16, 2]. Concrete examples of constant mean curvature surfaces are spheres, cylinders and Delauney surfaces.

Let u_1, u_2 be CMC graphs over $D_r(0)$ and assume that they have the same constant mean curvature ($H_{u_1} = H_{u_2}$), the same orientation ($\langle N_1, N_2\rangle > 0$), and that $u_1 - u_2 > 0$ then [2, Lemma 1.17] $u_1 - u_2 = v$ is a positive solution of

$$\operatorname{div} A_{ij}\nabla v + b\nabla v = 0 \tag{1.3}$$

where A_{ij} and b depend on ∇u_1 and ∇u_2. Moreover, if $|\nabla u_1|$ and $|\nabla u_2|$ are sufficiently small, we have the Harnack type inequality

$$\sup_{B_{\frac{r}{2}}(0)} (u_1 - u_2) \leq C_0(u_1(0) - u_2(0)). \tag{1.4}$$

Notice from Fig. 2 that the condition $\langle N_1, N_2 \rangle > 0$ on the orientation is necessary. As it is shown in Fig. 2 the two spherical caps have the same constant mean curvature since they have the same radius. However, even if $u_1(0) - u_2(0) = 0$, it is clear that $\sup(u_1 - u_2) > 0$ in any neighborhood of 0 and therefore that (1.4) does not follow. In

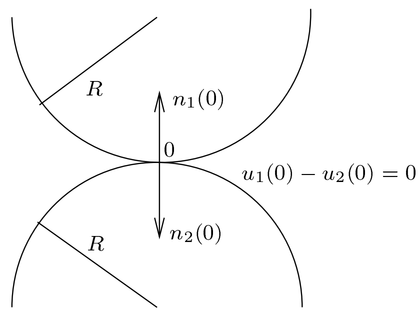

FIGURE 2

general, let k_1, k_2 be the principal curvatures on Σ, then $H = k_1 + k_2$; $|A|^2 = k_1^2 + k_2^2$ is the norm squared of the second fundamental form. Since the Gaussian curvature K_Σ is equal to the product of the principal curvatures $k_1 k_2$, we have the Gauss equation, that is

$$H^2 = |A|^2 + 2K_\Sigma. \tag{1.5}$$

From (1.5) it is clear why when H is constant, in particular when it is small and even better when it is zero, talking about the Gaussian curvature or talking about the norm of the second fundamental form squared is almost equivalent.

2. **Multi-valued graphs in CMC surfaces.**

In this section we discuss the result and explain the necessity of the hypotheses. We also take a closer look at how the proof goes.

This is what Colding and Minicozzi proved:

Theorem 2.1. [4, **Theorem 0.4.**] *Given $N \in \mathbb{Z}_+$, $\omega > 1$ and $\varepsilon > 0$, there exist $C = C(N, \omega, \varepsilon) > 0$ so:*
Let $0 \in \Sigma \subset B_R \subset \mathbb{R}^3$ be an embedded minimal disk such that $\partial\Sigma \subset B_R$. If

$$\sup_{\Sigma \cap B_{r_0}} |A|^2 \leq 4C^2 r_0^{-2} \text{ and } |A|^2(0) = C^2 r_0^{-2}$$

for some $0 < r_0 < R$, then there exists $\bar{R} < \frac{r_0}{\omega}$ and (after a rotation) an N-valued graph $\Sigma_g \subset \Sigma$ over $D_{\omega\bar{R}} \backslash D_{\bar{R}}$ with gradient $\leq \varepsilon$ and $\mathrm{dist}_\Sigma(0, \Sigma_g) \leq 4\bar{R}$.

Definition 2.2 (Multigraph)**.** *Let D_r be the disk in the plane centered at the origin and of radius r and let $\mathcal{P}$ be the universal cover of the punctured plane $\mathbb{C}\backslash 0$ with global coordinates (ρ, θ) so $\rho > 0$ and $\theta \in \mathbb{R}$. An N-valued graph of a function u on the annulus $D_s \backslash D_r$ is a single valued graph over $\{(\rho, \theta) | r \leq \rho \leq s, |\theta| \leq N\pi\}$.*

When dealing with multi-valued graphs, the surface to keep in mind is the helicoid, Fig. 3. A parametrization of the helicoid that illustrates the existence of such an N-valued graph is the following

$$(s\sin t, s\cos t, t) \qquad \text{where } (s,t)\in\mathbb{R}^2.$$

It is easy to see that it contains the N-valued graph ϕ defined by

$$\phi(\rho,\theta)=\theta \qquad \text{where } (\rho,\theta)\in\mathbb{R}^+\backslash 0\times[-N\pi,N\pi].$$

In fact the helicoid is a minimal surface. In Appendix A we provide examples of CMC

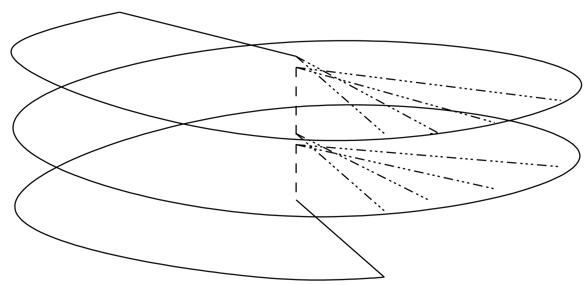

FIGURE 3. Half of the the Helicoids

surfaces containing arbitrary large multi-valued graphs.

What we are about to prove is not exactly Theorem 0.1. We prove the result when r_0 in Theorem 0.1 is equal to one and hence the curvature is bounded in a ball of radius $\bar{l}$. We will discuss and determine $\bar{l}$ in Section 5. Theorem 0.1 will follow by rescaling and we will describe the rescaling argument in Appendix B.

This is the new statement:

Theorem 2.3. *For each $N\in\mathbb{Z}_+$, $\omega>1$ and $\varepsilon>0$ there exist $H>0$, $C(N,\omega,\varepsilon)>0$ and $\bar{l}>1$ so:*
Let $0\in\Sigma\subset B_{\bar{l}}(0)\subset\mathbb{R}^3$ be an embedded and simply connected constant mean curvature surface equal to h (embedded CMC disk) such that $|h|\le H$ and $\partial\Sigma\subset\partial B_{\bar{l}}(0)$. If

$$\sup_{\Sigma\cap B_{\bar{l}}(0)}|A|^2\le 4C^2 \text{ and } |A|^2(0)=C^2$$

then there exists $\bar{R}<\frac{1}{\omega}$ and (after a rotation) an N-valued graph $\Sigma_g\subset\Sigma$ over $D_{\omega\bar{R}}\backslash D_{\bar{R}}$ (with gradient $\le\varepsilon$ and $\mathrm{dist}_\Sigma(0,\Sigma_g)\le 4\bar{R}$).

The constant $C(N,\omega,\varepsilon)$ is essentially the same constant that Colding and Minicozzi used.

We can only prove that a multi-valued graph exists substantially far away from the boundary, that is in a ball of radius one while the boundary of the surface is contained in the boundary of a ball of radius $\bar{l}>1$. For a minimal surface, Colding and Minicozzi were able to extend the multi-valued graph that forms locally, all the way up to the boundary [3]. It is not known if the same can be done for CMC surfaces.

Thanks to the upper bound on the second fundamental form, the surface is "uniformly locally flat" and the C^2 convergence follows. Moreover, $\sup_{\Sigma\cap B_{\bar{l}}(0)}|A|^2\le 4C^2$

together with the Gauss equation (1.5) gives a lower bound for the Gaussian curvature,

$$K_\Sigma \geq -4C^2 = 2G. \tag{2.1}$$

This lower bound implies an upper bound on the area of the intrinsic balls, Theorem 5.2.

What follows is a short sketch of the proof. The proof is by contradiction. Assuming that Theorem 2.3 is false we have the following:

> Given $C(N, \omega, \varepsilon)$ as in Theorem 2.1, for any $h > 0$ there exists an embedded and simply connected constant mean curvature equal h surface Σ_h that does not contain an N-valued graph $\Sigma_g \subset \Sigma$ over $D_{\omega \bar{R}} \backslash D_{\bar{R}}$ for any $\bar{R} < \frac{1}{\omega}$ but such that
>
> $$0 \in \Sigma \subset B_{\bar{l}}(0) \subset \mathbb{R}^3,\ \partial\Sigma \subset \partial B_{\bar{l}}(0) \text{ and } \sup_{\Sigma \cap B_{\bar{l}}} |A|^2 \leq 4C^2 = 4|A|^2(0).$$

We want to show that this cannot be true. Let us take a sequence of Σ_n as above with $h = \frac{1}{n}$. The constant mean curvature of Σ_n goes to zero but none of the elements in the sequence contain an N-valued graph. Fixed $\bar{\varepsilon} > 0$, we consider a new sequence Σ_n' where Σ_n' is the connected component of $\Sigma_n \cap B_{\bar{l}-\bar{\varepsilon}}(0)$ that contains 0. Given that $|A|^2$ is bounded and we are slightly away from the boundary there exists $r > 0$ so: Σ_n' can be covered by a finite number of balls, $B_r(x_i^n)$ where $x_i^n \in \Sigma_n'$, such that in each ball $\Sigma_n' \cap B_r(x_i^n)$ looks like graphs u_n^j over the tangent plane $T_{x_i^n}\Sigma_n$. The radius r and the number of balls will be independent of n. Going to a subsequence, we can assume that x_i^n converges to a certain x_i and that $T_{x_i^n}\Sigma_n$ converges to a certain $T_{x_i}\Sigma_\infty$. At this point we are able to extract, by using Arzela-Ascoli, a subsequence u_n^j that converges uniformly to a graph u_∞^j. These CMC graphs satisfy the following partial differential equation

$$\frac{1}{n} = \operatorname{div}\left(\frac{\nabla u_n^j}{\sqrt{1 + |\nabla u_n^j|^2}}\right).$$

Therefore, using Schauder theory [9] and the fact that $\frac{1}{n}$ goes to zero, we can prove that u_n^j converges C^2 to u_∞^j and that the latter is a minimal graph.

Unfortunately, we need more to prove the global properties required. The limit object contains a multi-valued graph if it is an embedded and simply connected minimal surface. We have not ruled out the possibility that the number j of graphs u_n^j goes to infinity as n goes to infinity and in the limit that could give an infinite number of minimal graphs. As a consequence the limit would not necessarily be a surface but it could be a lamination. Another possibility is that the limit is not simply connected, for instance it could be a catenoid, Fig. 4. Rescaling the catenoid in Fig. 4 the curvature at p becomes very large and yet the catenoid would not contain a multi-valued graph.

What we show is that the number of graphs is uniformly bounded if we stay substantially away from the boundary. This is because to prove this uniform upper-bound on the number of graphs we have to work with large geodesic balls and to assure that they exist, we need to move substantially away from the boundary. We need to be

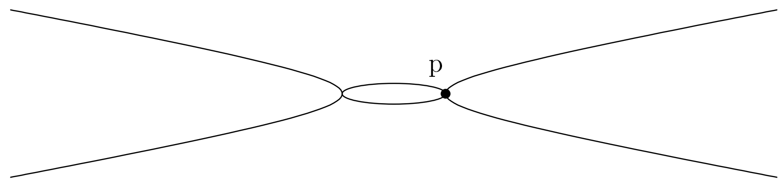

FIGURE 4. Catenoid

working in the unit ball and keep the boundary of the surface on a substantially bigger ball. More precisely, we build another subsequence Σ_n^1 where Σ_n^1 is the connected component of $\Sigma_n \cap B_1(0)$ that contains 0, Fig. 5. Σ_n^1 is also simply connected. If it

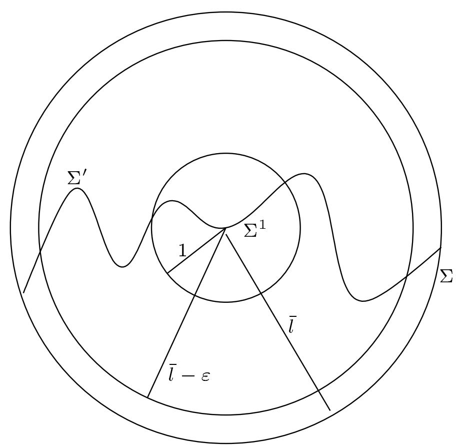

FIGURE 5

was not simply connected there would exist $B_r(0)$, $1 < r < \bar{l}$ such that Σ_n is tangent to $\partial B_r(0)$ and locally inside $B_r(0)$. This is a contradiction for $H(n) < \frac{1}{2\bar{l}}$, Fig. 6. If we restrict our attention to Σ_n^1 we have a uniform upper-bound on the number of

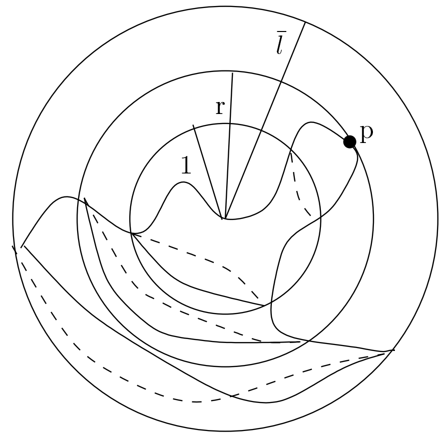

FIGURE 6

graphs and it follows that Σ_n^1, not the whole Σ_n, converges to an embedded minimal disk. Once we have that Σ^1_n converges to an embedded minimal disk, we prove that Σ^1_n, and therefore Σ_n, contains a multi-valued graph.

From now on, even if the results can often be stated more generally, Σ will be a CMC surface satisfying the hypotheses of Theorem 2.3. Σ' will be the connected component of $\Sigma \cap B_{\bar{l}-\bar{\varepsilon}}(0)$ containing 0. Σ^1 will be the connected component of $\Sigma \cap B_1(0)$ containing 0. We will also assume the mean curvature to be as small as we need, in particulary bounded.

3. δ-stability.

This section consists of standard results about CMC surfaces and stability.

Let A be the area functional described in Section 1; we showed that $A'(0) = \int_\Sigma \phi H$. A computation shows that if Σ is a CMC surface then

$$A''(0) = -\int_\Sigma \phi L_\Sigma \phi, \qquad \text{where } L_\Sigma \phi = \Delta_\Sigma \phi + |A|^2 \phi \tag{3.1}$$

is the second variational operator. Here Δ_Σ is the intrinsic Laplacian on Σ. A CMC surface Σ is said to be (strongly) stable if

$$A''(0) \geq 0 \qquad \text{for all } \phi \in C_0^\infty(\Sigma). \tag{3.2}$$

Applying Stokes' theorem to (3.2) shows that Σ is stable if and only if

$$\int_\Sigma |A|^2 \phi^2 \leq \int_\Sigma |\nabla \phi|^2, \qquad \text{for all } \phi \in C_0^\infty(\Sigma)$$

and that allows us to define δ-stability, namely Σ is said to be δ-stable if

$$(1-\delta)\int_\Sigma |A|^2 \phi^2 \leq \int_\Sigma |\nabla \phi|^2, \qquad \text{for all } \phi \in C_0^\infty(\Sigma). \tag{3.3}$$

In the following lemma we establish a relation between a CMC surface and a CMC normal variation of it that does not change the mean curvature.

Lemma 3.1. *There exists $\delta_1 > 0$ so: If $\delta < \delta_1$, Σ is a CMC surface and u is a positive solution of the CMC graph equation over Σ (i.e. $\Sigma^u := \{x + u(x)N_\Sigma(x) | x \in \Sigma\}$ is CMC) such that $|H_{\Sigma^u}| = |H_\Sigma|$, $\langle N_{\Sigma^u}, N_\Sigma \rangle \geq 0$, $|u||A|$ and $|\nabla u| \leq \delta$ then $\triangle u + u|A|^2 = o(\delta^2)$.*

Proof. In general

$$H_{\Sigma^u} = H_\Sigma + \frac{1}{2}(\triangle u + u|A|^2) + o(|u|^2, |\nabla u|^2).$$

The condition $\langle N_{\Sigma^u}, N_\Sigma \rangle \geq 0$ is a condition on the orientation that implies $H_{\Sigma^u} = H_\Sigma$ and the lemma follows. □

The existence of a positive solution of $Lu = 0$ where L is $\Delta + |A|^2$ would imply $A''(0) \geq 0$ for all $\phi \in C_0^\infty(\Sigma)$. In the following lemma we show that if there exists a positive function u which is "almost" a solution, then $A''(0)$ is "almost" non-negative for all $\phi \in C_0^\infty(\Sigma)$, that is, almost-stable.

Lemma 3.2. *Let Ω be a domain and u be a positive function in $C^2(\Omega)$ such that*

$$\Delta u \leq -(1-\delta)|A|^2 u \tag{3.4}$$

then Ω is δ-stable.

Proof. Set $w = \log u$ and let Φ be any compactly supported function on Ω. We have

$$\operatorname{div}(\nabla w) = \operatorname{div}(\frac{\nabla u}{u}) = \frac{\triangle u}{u} - \frac{|\nabla u|^2}{u^2} = \frac{\triangle u}{u} - |\nabla w|^2.$$

Applying Stokes theorem to $\operatorname{div}(\Phi^2 \nabla w)$ gives

$$0 = \int \operatorname{div}(\Phi^2 \nabla w) = \int \Phi^2 \Delta w + \int \langle \nabla \Phi^2, \nabla w \rangle.$$

Using Cauchy-Shwarz and the absorbing inequality gives

$$\int \langle \nabla \Phi^2, \nabla w \rangle \leq \int |\nabla \Phi|^2 + \int \Phi^2 |\nabla w|^2.$$

Eventually,

$$\int (-\frac{\triangle u}{u} + |\nabla w|^2)\Phi^2 \leq \int |\nabla \Phi|^2 + \int \Phi^2 |\nabla w|^2.$$

Applying (3.4) we get

$$(1-\delta) \int_\Sigma |A|^2 \phi^2 \leq \int_\Sigma |\nabla \phi|^2.$$

□

Lemma 3.1 and Lemma 3.2 give a first criteria to find almost stable domains in a constant mean curvature surface.

Corollary 3.3. *There exists $\delta_3 > 0$ so: If $\delta < \delta_3$, Σ is a CMC surface and u is a positive solution of the CMC graph equation over Σ such that $|H_{\Sigma^u}| = |H_\Sigma|$, $\langle N_{\Sigma^u}, N_\Sigma \rangle \geq 0$, $|u||A|$ and $|\nabla u| \leq \delta$ then Σ is δ-stable.*

4. **The upper bound on $|A|^2$.**

In this section we use the upper bound on $|A|^2$ to generalize some standard local results regarding CMC surfaces. We prove a criteria to find large pieces of Σ' which are graph over other pieces, creating large almost stable CMC domains.

Let us define

$$\Sigma_{x,R} \text{ as the component of } B_R(x) \cap \Sigma \text{ that contains } x, \tag{4.1}$$

$$\mathcal{B}_R(x) := \{y \in \Sigma \text{ such that } \operatorname{dist}_\Sigma(y, x) < R\} \tag{4.2}$$

i.e., the geodesic ball of radius R centered at x,

$$\mathcal{D}_r(x) := \{x' \in T_x\Sigma \text{ such that } |x - x'| < r\}. \tag{4.3}$$

In what follows we are about to explain why in a CMC surface with bounded $|A|^2$ everything looks graphical–what we have been calling "uniformly locally flat." Integrating $|\nabla \operatorname{dist}_{S^2}(\mathbf{n}(x), \mathbf{n})| \leq |A|$ on geodesics gives

$$\sup_{x' \in \mathcal{B}_s(x)} \operatorname{dist}_{S^2}(\mathbf{n}(x), \mathbf{n}) \leq s \sup_{\mathcal{B}_s(x)} |A|. \tag{4.4}$$

By (4.4), we can choose $0 < \rho < \frac{1}{4}$ so: If $\mathcal{B}_{2s}(x) \subset \Sigma$, $s \sup_{\mathcal{B}_s(x)} |A| \leq 4\rho_2$, and $t \leq s$ then the component $\Sigma_{x,t}$ of $B_t(x) \cap \Sigma$ with $x \in \Sigma_{x,t}$ is a graph over $T_x\Sigma$ with gradient $\leq \frac{t}{s}$ and

$$1 \geq \inf_{x' \in \mathcal{B}_2 s(x)} \frac{|x' - x|}{\operatorname{dist}_\Sigma(x, x')} > \frac{9}{10}. \tag{4.5}$$

One consequence is that if $t \leq s$ and we translate $T_x\Sigma$ so that $x \in T_x\Sigma$, then

$$\sup_{x' \in \mathcal{B}_t(x)} |x' - T_x\Sigma| \leq \frac{t^2}{s}. \tag{4.6}$$

As a consequence of (4.4), (4.5), (4.6) and the fact that in this paper we are assuming

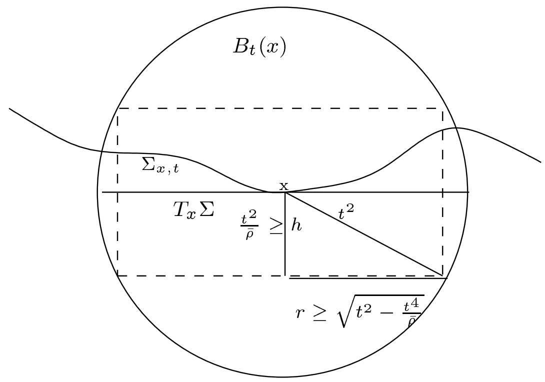

FIGURE 7

$\sup_\Sigma |A| < C$ we can clearly choose $0 < \bar\rho < 4\frac{\rho}{C}$ so: Given $t < \bar\rho$ and $x \in \Sigma$ then

$$\Sigma_{x,t} \text{ is a graph over } T_x\Sigma \text{ with gradient } \leq \frac{t}{\bar\rho} \text{ and } 1 \geq \inf_{x' \in \mathcal{B}_{2\bar\rho}(x)} \frac{|x' - x|}{\mathrm{dist}_\Sigma(x, x')} > \frac{9}{10}.$$

This means that, independently on x, $\Sigma_{x,t}$ is a graph over $T_x\Sigma$. Moreover, as shown in Fig. 7, using the Pythagorean theorem gives that

$$\text{the projection of } \Sigma_{x,t} \text{ onto } T_x\Sigma \text{ contains } \mathcal{D}_{\sqrt{t^2 - \frac{t^4}{\bar\rho^2}}}(x). \tag{4.7}$$

Furthermore, if $y \in B_t(x) \cap \Sigma$ and $\mathrm{dist}_\Sigma(x, y) \geq 2t$ then y cannot be in $\Sigma_{x,t}$, otherwise applying (4.5) gives

$$t \geq |y - x| > \frac{9}{10}\mathrm{dist}_\Sigma(x, y) \geq \frac{18}{10}t. \tag{4.8}$$

y is in a different component of $B_t(x) \cap \Sigma$. After defining an orientation y is either above or below $\Sigma_{x,t}$. For the same reason we can also add that $\mathcal{B}_{\bar\rho}(x) \cap \mathcal{B}_{\bar\rho}(y) = \emptyset$.

Corollary 3.3 tells us that under certain conditions regarding the orientation, if a CMC surface is a graph over another CMC surface with the same constant mean curvature, then it is almost stable. We are about to prove some lemmas which tell us when that happens and how large the almost stable domain is. This lemma shows how, if two pieces of Σ are close, then they must be graphs over the same plane.

Lemma 4.1. *There exists $\alpha_1 > 0$ so: For any $\alpha < \alpha_1$ and $x \in \Sigma'$ then any component of $B_\alpha(x) \cap \Sigma'$ is a graph over $T_x\Sigma$.*

Proof. Let us assume that there exists a component of $B_\alpha(x) \cap \Sigma'$ which is not a graph over $T_x\Sigma$. Then there exists $y \in B_\alpha(x) \cap \Sigma'$ such that $T_x\Sigma \perp T_y\Sigma$. If α_1 is small enough, it is clear from Fig. 8 that $\Sigma_{x,\frac{\bar\rho}{2}} \cap \Sigma_{y,\frac{\bar\rho}{2}} \neq \emptyset$ that is $y \in \mathcal{B}_{\bar\rho}(x)$. How we have chosen $\bar\rho$ implies that y must be part of a graph. Notice that we are also using the fact that we are slightly away from the boundary. $\Sigma_{x,\frac{\bar\rho}{2}} \cap \Sigma_{y,\frac{\bar\rho}{2}}$ could be empty if one

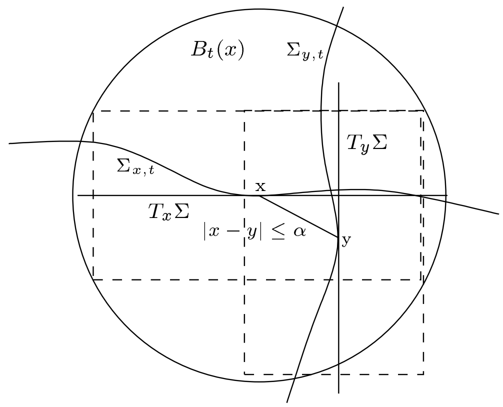

FIGURE 8

of the two sets reaches $\partial\Sigma$ before they intersect. How small α_1 must be will depend also on $\bar{\varepsilon}$.

□

In particular, it follows that if pieces of Σ' are very close then not only are they graphs over the same plane, they are graphs over each other. The idea is that if two graphs are almost flat over two different planes but they cannot intersect, then if the two graphs are close enough these two planes must have almost the same slope. One of the two graphs can therefore be seen as a graph over the other and this is what the next lemma is about.

Lemma 4.2. *There exists $\alpha_2 > 0, C_1 > 0$ and $s > 0$ so: Let $x, y \in \Sigma'$ such that $|x-y| \leq \alpha < \alpha_2$, $d_\Sigma(x,y) \geq 2\alpha$ and $\langle n(x), n(y)\rangle > 0$ then $\mathcal{B}_s(y)$ contains a graph $\{z + u(z)n(z)\}$ over a domain containing $\mathcal{B}_{\frac{s}{4}}(x)$, $|\nabla u| + |u| \leq \alpha C_1$.*

Proof. Assume $\alpha_2 < \alpha_1$. We know from (4.8) that y is in a different component of $B_\alpha(x) \cap \Sigma'$ and that $\mathcal{B}_{\bar{\rho}}(x) \cap \mathcal{B}_{\bar{\rho}}(y) = \emptyset$. If α_2 is sufficiently small, it now follows that $\Sigma_{x,\bar{\rho}}$ and $\Sigma_{y,\bar{\rho}}$ contain two graphs over the same plane, the smaller α_2 is the bigger the graph is. There exists $s > 0$ such that $\mathcal{B}_s(x)$ and $\mathcal{B}_s(y)$ contain respectively a graph u_1 and u_2 over $\mathcal{D}_{\frac{s}{2}}(x)$. $\langle n(x), n(y)\rangle > 0$ implies that the two intrinsic disks have equal constant mean curvature and (1.4) gives $\sup_{\mathcal{D}_{\frac{s}{4}}(x)} |u_1 - u_2| \leq \alpha C_0$. The function

$$u(x) = \min\{t \in \mathbb{R}_+ | x + tN(x) \in \mathcal{B}_s(y)\} \tag{4.9}$$

is well defined over $B_{\frac{s}{4}}(x)$ and $|\nabla u| + |u| \leq \alpha C_1$. □

Lemma 3.2 and Lemma 4.2 give a better criteria to find δ-stable domain.

Corollary 4.3. *Given $\delta > 0$ there exists $\alpha_3 > 0$ and $s > 0$ so: Let $x, y \in \Sigma'$ such that $|x-y| \leq \alpha < \alpha_3$, $d_\Sigma(x,y) \geq 2\alpha$ and $\langle n(x), n(y)\rangle > 0$ then $\mathcal{B}_{\frac{s}{4}}(x)$ is δ-stable.*

Proof. Let $\alpha_3 < \min(\frac{\delta}{C_1}, \alpha_2)$ and apply Lemma 3.2 and Lemma 4.2. □

In sum, we have proven that when two points $x, y \in \Sigma'$ are close enough to each other (Euclidean distance) and satisfy the condition on the orientation $\langle N(x), N(y)\rangle > 0$, then a little neighborhood of each point is δ-stable. We shall notice that the closer two pieces are the smaller δ is. The next step is to go from a little almost stable domain to a large one.

If we need a very large δ-stable geodesic ball, first of all we need the geodesic ball to be contained in Σ. In order to achieve this we certainly cannot be anywhere in the surface but sufficiently away from its boundary. If we move away from the boundary, as long as the objects we are working with are contained in Σ, thanks to the Harnack inequality we can find conditions that guarantee the existence of arbitrary large δ-stable domains. This is what we prove in the next lemma and corollaries. In Fig. 9 it is shown how if two pieces of Σ^1 are close then their extensions will have to stay relatively close.

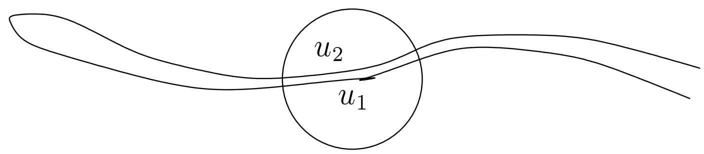

FIGURE 9

Lemma 4.4. *For each $0 < l < \bar{l} - (1 + \bar{\varepsilon})$ there exist $\alpha_l > 0$ and $C_l > 0$ so: Given $\alpha < \alpha_l$ and $x, y \in \Sigma^1$ such that $|x - y| \leq \alpha$, $d_\Sigma(x, y) \geq C_l$ and $\langle n(x), n(y)\rangle > 0$ then for each $x' \in \mathcal{B}_l(x)$ there exists $y' \in \Sigma'$ such that $|x' - y'| \leq \alpha_1$, $d_\Sigma(x', y') \geq 2\alpha_1$ and $\langle n(x'), n(y')\rangle > 0$.*

Proof. Fix $N \in \mathbb{N}$ such that $\frac{4l}{s} \leq N < \frac{4l}{s} + 1$, s as in Corollary 4.3 and assume $\alpha_l < \min(\alpha_2, \frac{\alpha_2}{C_0^N})$, C_0 being the constant as in (1.4). Our goal is to find C_l, note that $\mathcal{B}_l(x) \subset \Sigma'$. Let $x' \in \mathcal{B}_l(x)$ then there exists a geodesic $\gamma(t)$, $t \in [0, 1]$ such that $\gamma(0) = x$, $\gamma(1) = x'$ and length$(\gamma) \leq l$. Fix a partition $\mathcal{Q}$ of $[0, 1]$, $\mathcal{Q} = \{t_i \in [0, 1] | 0 \leq i \leq T\}$, such that

$$\begin{cases} t_0 = 0, t_T = 1 \\ d_\Sigma(\gamma(t_i) = x_i, \gamma(t_{i+1}) = x_{i+1}) \leq \frac{s}{4} \\ T \leq N \end{cases} . \tag{4.10}$$

Since $|x - y| \leq \alpha \leq \alpha_2$ and $d_\Sigma(x, y) \geq C_l \geq 2\alpha_2$ then Lemma 4.2 gives $\mathcal{B}_s(x)$ and $\mathcal{B}_s(y)$ contain respectively a graph u_1 and u_2 over $\mathcal{D}_{\frac{s}{2}(x)}$ and $\sup_{\mathcal{D}_{\frac{s}{4}(x)}} |u_1 - u_2| \leq C_0\alpha$. Therefore, let $z_1 \in \mathcal{D}_{\frac{s}{4}}(x)$ such that $x_1 = u_1(z_1)$ and let $y_1 = u_2(z_1)$, then $|x_1 - y_1| \leq C_0\alpha \leq \alpha_2$, $d_\Sigma(x_1, y_1) \geq d_\Sigma(x, y) - d_\Sigma(x_1, x) - d_\Sigma(y_1, y) \geq C_l - \frac{s}{4} - \frac{5s}{4} \geq 2\alpha_2$ if C_l is big enough, and $\langle N(x_1), N(y_1)\rangle > 0$. As long as $d_\Sigma(x_i, y_i) \geq 2\alpha_2$ we can apply Lemma 4.2. We can repeat this argument N times as long as $C_l - N\frac{3s}{2} \geq 2\alpha_2$. □

Corollary 4.5. *For each $0 < l < \bar{l} - (1 + \bar{\varepsilon})$ there exist $\alpha_l > 0$ and $C_l > 0$ so: Given $\alpha < \alpha_l$ and $x, y \in \Sigma^1$ such that $|x - y| \leq \alpha$, $d_\Sigma(x, y) \geq C_l$ and $< n(x), n(y) >> 0$ then for each $x' \in \mathcal{B}_l(x)$, $\mathcal{B}_t(y)$ contains a graph $\{z + u(z)n(z)\}$ over a domain containing $\mathcal{B}_{\frac{t}{4}}(x)$, $|\nabla u| + |u| \leq \alpha C_1$.*

Proof. Apply Lemma 4.4 and Lemma 4.2. □

Corollary 4.6. *For each $0 < l < \bar{l} - (1 + \bar{epsilon})$ and $\delta > 0$ there exist $\alpha_{l,\delta} > 0$ and $C_l > 0$ so: Given $\alpha < \alpha_{l,\delta}$ and $x, y \in \Sigma^1$ such that $|x - y| \leq \alpha$, $d_\Sigma(x, y) \geq C_l$ and $\langle N(x), N(y) \rangle > 0$ then $\mathcal{B}_l(x)$ is δ-stable.*

Proof. Let $\alpha_{l,\delta} < \min(\alpha_l, \frac{\delta}{C_1})$, according to Corollary 3.3, we need to find a u which is a positive solution of the CMC graph equation over $\mathcal{B}_l(x)$. Corollary 4.5 gives that the latter is true locally. In fact, fix $x_i \in \mathcal{B}_l(x)$ such that $\mathcal{B}_s(x_i)$ and $\mathcal{B}_{\frac{s}{4}}(x_i)$ are both finite coverings for $\mathcal{B}_l(x)$. From Lemma 4.4 it follows that $\mathcal{B}_s(x_i)$ contains a graph $\{z + u_i(z)n(z)\}$ over a domain containing $\mathcal{B}_{\frac{s}{4}}(x_i)$, $|\nabla u_i| + |u_i| \leq \alpha C_1$. The function $u(y) := u_i(y)$ if $y \in \mathcal{B}_{\frac{s}{4}}(x_i)$ is a well defined function over $\mathcal{B}_l(x)$ such that $|\nabla u| + |u| \leq \alpha C_1 < \delta$. Applying Corollary 3.3 gives this corollary. □

5. The non-existence of large almost stable domain and the uniform bound.

We have seen when it happens that Σ contains a large almost stable domain. In this section we show that an almost stable domain cannot be too large. Using these two facts together we prove a uniform bound on the number of graphs.

In order to continue with this proof by contradiction we state the following result by Sirong Zhang [18, **Theorem 0.1.**], and the Bishop Volume Comparison Theorem [17, **Theorem 1.3.**]:

Theorem 5.1. *There exists a C such that given any $l > 0$ there exists an $h > 0$ so: If $\mathcal{B}_l(0)$ is a "constant mean curvature equal to h", δ-stable intrinsic disk with trivial normal bundle then* $\sup_{\mathcal{B}_{\frac{l}{2}}(0)} |A|^2 \leq \frac{C}{l^2}$.

Theorem 5.2 (Bishop Volume Comparison Theorem)**.** *Let M be an n-dimensional complete Riemannian manifold with $Ric(M) \geq (n-1)K$. Then for any $x \in M$ and $R > 0$, $\frac{Vol(\mathrm{B}_R(x))}{V(K,R)}$ is a non-increasing function in R. Hence,*

$$Vol(\mathrm{B}_R(x)) \leq V(K, R),$$

where $V(K, R)$ is the volume of the geodesic ball of radius R in the space form M_K.

Theorem 5.1 can be thought as a generalization of [7, 8] and we will see how it is essentially what determines how big $\bar{l}$ is. Our surface has trivial normal bundle since it is orientable and this will be proved later in Theorem 6.4. Theorem 5.2 gives that for any $x \in \Sigma$

$$Vol(B_R(x)) \leq V(G, R), \qquad G \text{ as in 2.1.} \tag{5.1}$$

The following proposition uses what we have proved in the previous sections and Theorem 5.1 to show that Σ' does not contain a large almost stable domain. Roughly speaking if we take l large and assume that $\mathcal{B}_l(x) \subset \Sigma^1$ is δ-stable, Theorem 5.1 implies that $\mathcal{B}_{\frac{l}{2}}(x)$ is almost flat. This forces the intrinsic disk to leave the unit ball.

Proposition 5.3. *Given $\delta > 0$ there exists $l_\delta > 0$ so: For any $l \geq l_\delta$ and $x \in \Sigma^1$ if $\mathcal{B}_l(x)$ is δ-stable then it is not contained in Σ^1.*

Proof. Let us fix $l_\delta > \max(\frac{20}{9}, \sqrt{\frac{C}{\rho}})$. Let C be as in Theorem 5.1 and ρ as in (4.5). Being $\mathcal{B}_l(x)$ δ-stable Theorem 5.1 implies

$$\sup_{\mathcal{B}_{\frac{l}{2}}(x)} |A|^2 \leq \frac{C}{l^2} \leq \rho.$$

(4.5) implies that

$$\inf_{\mathcal{B}_{\frac{l}{2}}(x)} \frac{|x-y|}{d_\Sigma(x,y)} \geq \frac{9}{10}.$$

Taking $y \in \mathcal{B}_{\frac{l}{2}}(x)$ such that $d_\Sigma(x,y) > \frac{10}{9}$, we have $|x-y| > 1$. This proves that $\mathcal{B}_l(x)$ is not contained in Σ^1. □

We have proved so far that decreasing the Euclidean distance between two points gives a large δ-stable domain as long as we increase their intrinsic distance. In the following lemma we apply the Bishop Volume Comparison Theorem and a lower bound on the area of each piece to prove that the more graphs there are in a small ball the larger the intrinsic distance becomes. This is what Fig. 10 illustrates.

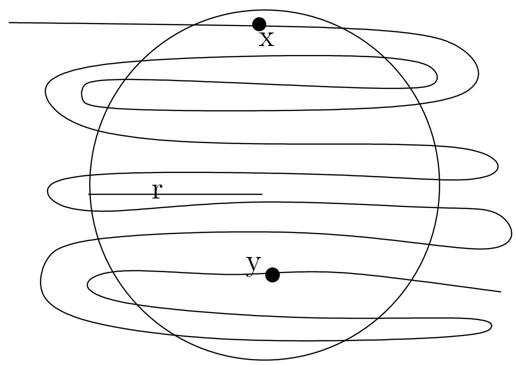

FIGURE 10

Let us fix $\delta, \bar{\varepsilon} > 0$ small, let $l_1 = l_\delta$ as given by Proposition 5.3 and let $\bar{l}$ in Theorem 2.3 to be equal to $l_1 + 1 + \bar{\varepsilon}$. In other words,

Σ^1 does not contain a δ-stable geodesic ball of radius bigger than l_1.

Let us fix $0 < r < \alpha_{l_1,\delta}$ where $\alpha_{l_1,\delta}$ is taken as in Corollary 4.6. This means that

$$\text{For any } x \in \Sigma', \ B_r(x) \cap \Sigma' \text{ consists of graphs over } T_x\Sigma \tag{5.2}$$

and also that

$$\begin{aligned} &\text{There exists } C_{l_1} > 0 \text{ such that if } x, y \in B_r(x) \cap \Sigma^1,\ d_\Sigma(x,y) \geq C_{l_1} \\ &\text{and } \langle N(x), N(y)\rangle > 0 \text{ then } \mathcal{B}_{l_1}(x) \text{ is } \delta\text{-stable.}\end{aligned} \tag{5.3}$$

Given $x \in \Sigma^1$ let n_x be the number of components of $B_r(x) \cap \Sigma^1$. The area of each component Σ_i^x could go to zero if they accumulate toward the boundary of the ball. Nonetheless we have proven that these graphical pieces continue outside the ball, 4.7. Therefore we have a uniform lower bound on the area of Σ_i^x; in other words

$$\text{There exits } \varepsilon > 0 \text{ such that } \mathrm{Area}(\Sigma_i^x) > \varepsilon \text{ independently of } x \text{ and } i. \tag{5.4}$$

Lemma 5.4. *Given $\lambda > 0$ there exists $n_\lambda > 0$ so: If $x \in \Sigma^1$ and $n_x > n_\lambda$ then there exist $y, y' \in \mathrm{B}_r \cap \Sigma^1$ such that $dist_\Sigma(y, y') > \lambda$ and $\langle n(y), n(y')\rangle > 0$.*

Proof. Theorem 5.2, that is, Bishop Volume Comparison Theorem, gives an upper bound for the area of $\mathcal{B}_\lambda(x)$, namely

$$\mathrm{Area}(\mathcal{B}_\lambda(x)) < V(G, \lambda).$$

At this point it follows easily that if $n_x \in \mathbb{N}$, $n_x > \frac{V(G,\lambda)}{\varepsilon}$ then there exists $y_1 \in B_r \cap \Sigma^1$ which is not in $\mathcal{B}_\lambda(x)$, i.e. $\mathrm{dist}_\Sigma(x, y_1) > \lambda$. $\langle n(x), n(y_1)\rangle > 0$ does not necessarily happen. Take $\bar{V} \in \mathbb{N}$ such that $\bar{V} - 1 < \frac{V(G,\lambda)}{\varepsilon} \leq \bar{V}$ and let $n_\lambda = \bar{V}^{\bar{V}}$. If $n_x \geq n_\lambda$ then there exist at least $\bar{V}$ distinct $y_i,\ i := 1, ...\bar{K}$, in different component of $B_r(x) \cap \Sigma^1$ such that $\mathrm{dist}_\Sigma(x, y_i) > \lambda$. Fixed y_1 there exists y_2 among the y_i such that $\mathrm{dist}_\Sigma(y_1, y_2) > \lambda$. At this point either $\langle n(x), n(y_1)\rangle > 0$ or $\langle n(x), n(y_2)\rangle > 0$ or $\langle n(y_1), n(y_2)\rangle > 0$. □

The following corollary uses Proposition 5.3 and Lemma 5.4 to obtain the upper bound on the number of graphs.

Corollary 5.5. *For any $x \in \Sigma^1$, $n_x \leq n_{C_{l_1}}$.*

Proof. If $n_x > n_{C_{l_1}}$ then Lemma 5.4 with λ equal to C_{l_1} gives that there exist $y, y' \in B_r(x) \cap \Sigma^1$ such that $\mathrm{dist}_\Sigma(y, y') > C_{l_1}$ and $\langle n(y), n(y')\rangle > 0$. Using Lemma 4.6 gives that $\mathcal{B}_{l_1}(x)$ is δ-stable and, by Proposition 5.3, cannot be contained in Σ^1. Since $\mathrm{dist}_\Sigma(y, y') > C_{l_1} > l_1$, y' is not in Σ^1. This gives the contradiction that implies $n_x \leq n_{C_{l_1}}$. □

6. **Multi-valued graphs in CMC surfaces.**

In this final section we show that the Σ^1_n converges C^2 to an embedded minimal disk Σ_∞ that contains a multi-valued graph. It follows that the CMC surfaces in the sequence contain a multi-valued graph as well. The limit surface is embedded and minimal by a standard argument which will be sketched below. To prove that it is simply connected we need more work and well-known topological results.

Let $r > 0$ be as defined in (5.2) and (5.3), that is, there exists a finite covering $B_r(x^n_i)$ for Σ^1_n where everything is graphical over $T_{x^n_i}\Sigma_n$ and the number of graphs is uniformly bounded. We can also assume that the number of balls involved is uniformly bounded with respect to n. Going to a subsequence, we can assume x^n_i converging to x_i and $T_{x^n_i}\Sigma_n$ converging to a certain $T_{x_i}\Sigma_\infty$. Using the argument outlined in Section 2, the fact that the number of graphs is uniformly bounded and the maximum principle for minimal surfaces gives that the limit is an embedded minimal surface. Fig. 11 illustrates the two types of intersection that could occur if

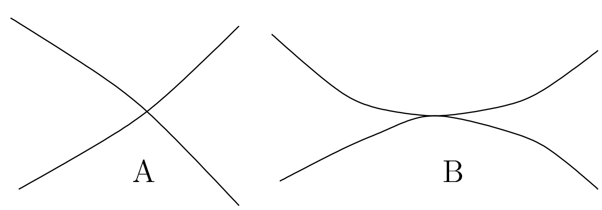

FIGURE 11

the limit is not embedded: A cross intersection, type A, and a tangential intersection, type B. However type A cannot be a continuous limit of embedded surfaces. Type B, which could be the limit of a sequence of embedded surfaces, cannot occur because of the maximum principle for minimal surfaces. By continuity the curvature of this minimal surface is large at zero.

To prove that Σ_∞ is simply connected we use some results about Jordan curves [10] and the following theorems:

Theorem 6.1. [10, **Chapter 3**] *Every compact hypersurface in Euclidean space without boundary is orientable.*

Theorem 6.2. [10, **Chapter 3**] *If M is an orientable surface, so is M minus one point.*

Theorem 6.3. [11, **Corollary 3.28.**] *If M is a closed connected n-manifold, the torsion subgroup of $H_{n-1}(M, \mathbb{Z})$ is trivial if M is orientable and $\mathbb{Z}_2$ if M is not orientable.*

In our case Σ_∞ is a closed connected 2-manifold, and it is a consequence of Theorem 6.3 that, if it is orientable, the torsion subgroup of its fundamental group is trivial. The next proposition shows that Σ_∞ is the embedded minimal disk we have been looking for.

Proposition 6.4. *Σ_∞ is an embedded simply connected minimal surface such that, $0 \in \Sigma_\infty \subset B_1 \subset \mathbb{R}^3$, $\partial\Sigma_\infty \subset \partial B_1$, $\sup_{\Sigma\cap B_1} |A|^2 \leq 4C^2$ and $|A|^2(0) = C^2$.*

Proof. The conditions on the second fundamental form, that is

$$\sup_{\Sigma\cap B_1} |A|^2 \leq 4C^2 \text{ and } |A|^2(0) = C^2,$$

are a consequence of the C^2 convergence. We have already proved that it is embedded and what we are left to prove is that Σ_∞ is simply connected. Let us prove that it is orientable first. We want to prove that Σ_∞ is homeomorphic to a compact embedded surface minus a finite number of points. Because it is an embedded minimal surface, $\partial\Sigma_\infty$ is a finite number of disjoint loops ξ_i, $i := 1, ..., I$ which do not have self intersections, Fig. 12. These loops lie on $\partial B_1(0)$ minus one point, hence we have

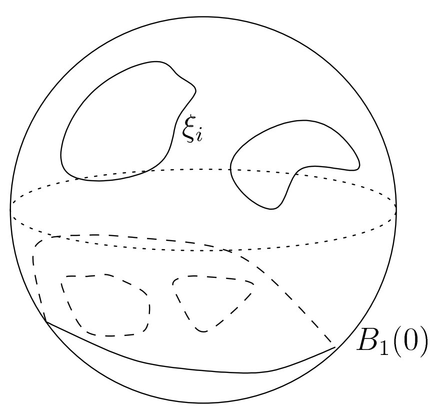

FIGURE 12

essentially a finite number of Jordan curves in the plane. We can glue I disks to

Σ_∞ in a way that the result is an embedded compact surface: Each Jordan curve ξ_i divides the plane into an inner and an outer region, and can be thought of as the boundary of a simply connected domain, namely a disk D_i. If a loop ξ_i lies in the inside of another loop ξ_j then we lift D_j so that it does not intersect D_i. Fig. 13 illustrates how we are gluing these disks to the surface. Since the number of loops is

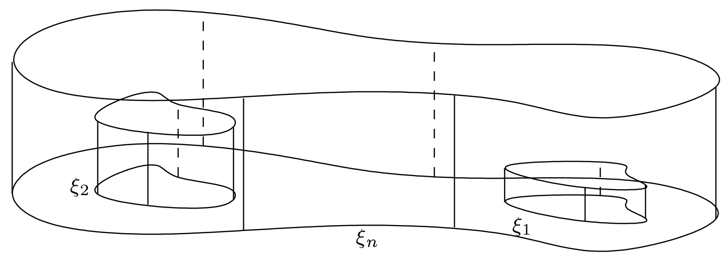

Figure 13

finite we repeat this a finite number of times and obtain in the end a new surface

$$\bar{\Sigma} = \{\Sigma_\infty \bigcup_{i=1}^{I} D_i\}/\sim$$

where $\sim$ is the relation that identify ξ_i with ∂D_i. $\bar{\Sigma}$ is a compact embedded surface without boundary and therefore orientable by Theorem 6.1. Theorem 6.2 implies that $\bar{\Sigma}$ take out a finite number of points is still orientable, that is $\bar{\Sigma}\setminus\bigcup_{i=1}^{I} D_i = \Sigma_\infty$. Theorem 6.3 tells that $\pi_1(\Sigma_\infty)$ is torsion free.

Let $\gamma: S^1 \longrightarrow \Sigma_\infty$ be a closed path and $B_\sigma(\gamma(t_i))$ a finite covering for γ such that $\sigma < \min(1-|\gamma|, r)$, and the number of components of $B_\sigma(\gamma(t))$ is non-increasing. This is possible after going to a subsequence, assuming n large because of the uniform bound. Fix a starting point $\gamma(t_0)$, an orientation on γ, and let

$$\gamma_n(t_0) := \gamma(t_0) + s_0 N_{\gamma(t_0)} \in \Sigma_n^1.$$

Moving continuously on $\gamma(t)$ we obtain a new path

$$\gamma_n(t) := \gamma(t) + s(t) N_{\gamma(t)} \in \Sigma_n^1.$$

The conditions on σ and n force the path to close up after it moves around γ a finite number of times, $k \in \mathbb{N}$. Since Σ_n^1 simply connected, there exists a map $\Gamma_n: D_1 \longrightarrow \Sigma_n^1$ such that $\Gamma_n|_{S^1} = \gamma_n$. Define $\Gamma: D_1 \longrightarrow \Sigma_\infty$,

$$\Gamma(x) := \lim_{n\longrightarrow\infty} \Gamma_n(x).$$

The existence of this map Γ proves that $k\gamma$ is homotopic to a point. Since $\pi_1(\Sigma_\infty)$ is torsion free, this implies that γ itself is homotopic to a point. Since γ could be any path on Σ_∞ we have proved that Σ_∞ is simply connected. $\square$

Finally we prove that Σ_n^1 and therefore Σ_n contains a multi-valued graph. In Proposition 6.4 we proved that Σ_∞ is an embedded simply connected minimal surface such that,

$$0 \in \Sigma_\infty \subset B_1 \subset \mathbb{R}^3, \partial\Sigma_\infty \subset \partial B_1, \sup_{\Sigma\cap B_1} |A|^2 \leq 4C^2 = 4|A|^2(0).$$

Taking $C = C(N, \omega, \varepsilon)$ as in Theorem 2.1, the same theorem gives that Σ_∞ contains and N-valued graph. Let u be this N-valued graph, defined over $\{(\rho,\theta)|r \leq \rho \leq s, |\theta| \leq N\pi\}$ as described in Definition 2.2. This is how we build an N-valued graph in Σ_n: Given $r \leq \bar\rho \leq s$, define $u_{\bar\rho}(\theta) = u(\bar\rho,\theta)$. Consider $u_{\bar\rho}$ as a path on Σ_∞ starting at $u_{\bar\rho}(-N\pi)$. Assuming n large, Σ^1_n moves closer and closer to Σ_∞ and there exists a continuous function $\phi(\theta)$ such that

$$u^n_{\bar\rho}(\theta) = u_{\bar\rho}(\theta) + \phi(\theta)\mathbf{e}_3 \in \Sigma^1_n$$

is well defined. The function $u^n(\rho,\theta) = u^n_\rho(\theta)$ defined over $\{(\rho,\theta)|r \leq \rho \leq s, |\theta| \leq N\pi\}$ is an N-valued graph.

Notice that as Σ^1_n moves closer and closer to Σ_∞, Σ^1_n and Σ_∞ are "parallel surfaces." Not only does Σ_n contain an N-valued graph, but the properties of this graph, such as the upper bound on the gradient, are preserved.

APPENDIX A

In this appendix we provide examples of CMC surfaces containing arbitrary large multi-valued graphs. We use the method of successive approximations to build a sequence of normal variations of the helicoid that converges to an embedded and simply connected CMC surface containing a multi-valued graph.

Let $\Sigma_h = \{x + u(x)N_\Sigma(x), x \in \Sigma\}$ be a normal variation of Σ, where Σ is any minimal surface. Σ_h is a CMC surface with mean curvature equal to H if $u(x)$ satisfies the following equation [12, 13, 15]:

$$Lu = H + Q(u), \qquad \text{where } Lu = \Delta u + |A|^2 u \tag{A-1}$$

is the linearized operator. Q is a quadratic and higher order function in u, u_i, u_{ij} where $i,\ j \in \{1,2\}$, with geometric invariants of Σ as coefficients. Before we prove the existence of a constant mean curvature normal variation of the helicoid we need to describe some properties of the function Q.

Let $C^{k,\lambda}(\Sigma)$ be the standard subset of $C^k(\Sigma)$ consisting of functions whose k-th partial derivatives are Hölder continuous with exponent λ in Σ and let $\|\cdot\|_{k,\lambda}$ be the notation for the Hölder norm. Let us define Ω_δ, subset of $C^{2,\lambda}(\Sigma)$, in the following way:

$$\Omega_\delta := \{u \in C^{2,\lambda} : \|u\|_{2,\lambda} < \delta\}. \tag{A-2}$$

The following lemma follows from [12, Lemma C.2] and it is a consequence of the fact that Q is a quadratic and higher order function. It says that $\|Q(u)\|_{0,\lambda}$ decays faster than $\|u\|_2\|u\|_{2,\lambda}$.

Lemma A-1. *There exist $\delta_1 > 0$ and $C_1 > 0$ so: If $u \in C^{2,\lambda}(\Sigma)$ and $|A||u|, |u_i| < \delta_1$ then*

$$\|Q(u)\|_{0,\lambda} < C_1\|u\|_2\|u\|_{2,\lambda}.$$

As a consequence of Lemma A-1, we have a new corollary that relates $\|Q(u)\|_{0,\lambda}$ and $\|u\|_{2,\lambda}$:

Corollary A-2. *Given $1 > C_2 > 0$ there exists $\delta_2 > 0$ so: If $u \in \Omega_{\delta_2}$ and $|A||u| < \delta_2$ then*

$$\|Q(u)\|_{0,\lambda} < C_2\|u\|_{2,\lambda}.$$

Proof. Let $\delta_2 < \min(\frac{C_2}{C_1}, \delta_1)$. This implies that $C_1\|u\|_2 < C_1\|u\|_{2,\lambda} < C_1\delta_2 < C_2$ and that $|A||u|, |u_i| < \delta_2 < \delta_1$. Therefore, we can apply A-1 and we have

$$\|Q(u)\|_{0,\lambda} < C_1\|u\|_2\|u\|_{2,\lambda} < C_2\|u\|_{2,\lambda}.$$

□

Let Σ be a simply connected disk in the helicoid that contains a multi-valued graph. Due to the domain monotonicity and continuity of eigenvalues [1] we can also assume that 0 is not an eigenvalue for L on Σ and therefore that the Dirichlet

$$\begin{cases} Lu = w \\ u|_{\partial\Sigma} = 0 \end{cases} \tag{A-3}$$

has a unique solution $u \in C^{2,\lambda}(\Sigma)$ for any $w \in C^{0,\lambda}(\Sigma)$. Assuming that 0 is not an eigenvalue for L gives also the following lemma [9, Theorem 5.3 and page 109]:

Lemma A-3. *There exists a constant B depending only on Σ so: Let $u \in C^{2,\lambda}(\Sigma)$ be the unique solution for* (A-3) *then*

$$\|u\|_{2,\lambda} < B\|w\|_{0,\lambda}. \tag{A-4}$$

We will prove that there exists $H > 0$ such that a solution u for the Dirichlet problem

$$\begin{cases} Lu = H + Q(u) \\ u|_{\partial\Sigma} = 0 \end{cases} \tag{A-5}$$

exists and $\|u\|_{L^\infty}$ is small. The existence of a fixed neighborhood of the helicoid where the normal exponential map is injective guarantees that $\Sigma_H = \{x+u(x)N_\Sigma(x), x \in \Sigma\}$ is embedded, if $\|u\|_{L^\infty}$ is small enough. What we are about to show is that if H is small enough we can build a sequence of normal variations u^n of the helicoid that converges to a CMC normal variation u. We will also show that $\|u\|_{L^\infty}$ can be as small as we want and consequently that the CMC normal variation is embedded.

Let u^1 be the unique solution for

$$\begin{cases} Lu^1 = H \\ u^1|_{\partial\Sigma} = 0 \end{cases} \tag{A-6}$$

and u^n be the unique solution for

$$\begin{cases} Lu^n = H + Q(u^{n-1}) \\ u^n|_{\partial\Sigma} = 0 \end{cases} . \tag{A-7}$$

Lemma A-3 implies that

$$\|u^1\|_{2,\lambda} < BH$$

and also that

$$\|u^k\|_{2,\lambda} < B(H + \|Q(u^{k-1})\|_{0,\lambda}). \tag{A-8}$$

The existence of a solution for

$$Lu = H + Q(u)$$

will follow clearly, and we will see how, applying Arzela-Ascoli to the sequence u^n if we prove that there exists a constant K such that $\|u^n\|_{2,\lambda} < K$ uniformly in n. Fix C_2 in Corollary A-2 so that $\varepsilon = C_2 B < 1$, B as in Lemma A-3. We will prove by strong induction that

if H is so that $BH(1+\frac{1}{1-\varepsilon}) < \delta_2$ then $u^n \in \Omega_{\delta_2}$ for any n,

that is what we wanted. We have already that

$$\|u^1\|_{2,\lambda} < BH < \delta_2,$$

namely the statement is true for $n = 1$. Let us prove that

"true for $n = 1$ implies true for $n = 2$."

Assuming $\|u^1\|_{2,\lambda} < \delta_2$, we can apply Lemma A-1 that gives that $\|Q(u^1)\|_{0,\lambda} < C_2\|u^1\|_{2,\lambda} < \varepsilon H$, therefore

$$\|u^2\|_{2,\lambda} \le B(H + \|Q(u^1)\|_{0,\lambda} \le B(H + \varepsilon H) < \delta_2. \tag{A-9}$$

Let us prove that

"true for all k with $k \le n$ implies true for $k = n + 1$."

"True for all k with $k \le n$" means that $\|u^k\|_{2,\lambda} < \delta_2$ for $k \le n$ and therefore Lemma A-1 gives that

$$\|Q(u^k)\|_{0,\lambda} < C_2\|u^k\|_{2,\lambda} < C_2 B(H + \|Q(u^{k-1})\|_{0,\lambda}) < \varepsilon(H + \|Q(u^{k-1})\|_{0,\lambda}) \tag{A-10}$$

for $k \le n$. Applying (A-10) n times we have

$$\begin{aligned} \|u^{n+1}\|_{2,\lambda} &< B(H + \|Q(u^n)\|_{0,\lambda}) < B(H + \varepsilon(H + \|Q(u^{n-1})\|_{0,\lambda})) < \\ &< B(H + \varepsilon(H + \varepsilon(H + \|Q(u^{n-2})\|_{0,\lambda}))) < \\ &< BH(1 + \sum_{k=1}^{n} \varepsilon^k) < BH(1 + \frac{1}{1-\varepsilon}) < \delta_2. \end{aligned} \tag{A-11}$$

Now that we have proved that $\|u^n\|_{2,\lambda} < \delta_2$ uniformly in n, using Arzela-Ascoli we can extract a subsequence that converges C^2 to a certain $u \in C^2(\Sigma)$. Taking the limit as n goes to infinity on both sides of the equation

$$Lu^n = H + Q(u^{n-1})$$

gives that

$$Lu = H + Q(u).$$

u is therefore a constant mean curvature normal variation of the helicoid. It is clear from the proof that taking H small gives $\|u\|_{L^\infty}$ small. Consequently, the constant mean curvature normal variation that we have built is also embedded.

APPENDIX B

In this appendix we want to show that Theorem 0.1 follows from Theorem 2.3 by rescaling. The result is true even when the mean curvature is large but on a smaller ball. In other words, surfaces with large constant mean curvature have tiny multi-valued graphs around the origin. We give the idea of how that happens when dealing with simple graphs.

Let us assume that the CMC surface Σ is given as a graph u over the unit disk (given that it contains a multi-valued graph this cannot actually happen globally but it is always possible locally and we are assuming it now just for the sake of simplicity). Then the new surface Σ' given by $w(x) = Ru(\frac{x}{R})$ defined over $D_R(0)$ is still a CMC surface. In fact

$$H(\Sigma') = \operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right) = \frac{1}{R}H(\Sigma). \tag{B-1}$$

Therefore, assuming for instance $R < 1$ the new CMC surface has bigger mean curvature and the multi-graph happens in a smaller ball (very tiny if H is big). Rescaling preserves the existence of the multi-valued graph but changes the hypotheses regarding $|A|^2$. Since $k_i' = \frac{1}{R}k_i$, we have $|A'|^2 = \frac{|A|^2}{R^2}$ and therefore

$$|A'(0)|^2 = \frac{|A(0)|^2}{R^2} = \frac{C^2}{R^2} \text{ and } \sup_{\Sigma' \cap B_R} |A'|^2 = \frac{\sup_{\Sigma \cap B_1} |A|^2}{R^2} \leq \frac{4}{R^2}C^2$$

hence Theorem 0.1.

References

[1] I. Chavel: *Eigenvalues in Riemannian Geometry*, Academic Press, INC., Orlando, Florida, 1984.

[2] T.H. Colding and W.P. Minicozzi II: *Minimal Surfaces*, Courant Lecture Notes in Math., v. 4, 1999.

[3] T.H. Colding and W.P. Minicozzi, *The space of emedded minimal surfaces of fixed genus in a 3-manifold I; Estimates off the axis for disks*, Annals of Math, to appear, math.AP/0210106.

[4] T.H. Colding and W.P. Minicozzi, *The space of emedded minimal surfaces of fixed genus in a 3-manifold II; Multi-valued graphs in a disk*, Annals of Math, to appear, math.AP/02100086.

[5] T.H. Colding and W.P. Minicozzi, *The space of emedded minimal surfaces of fixed genus in a 3-manifold III; Planar domains*, Annals of Math, to appear, math.AP/0210141.

[6] T.H. Colding and W.P. Minicozzi, *The space of emedded minimal surfaces of fixed genus in a 3-manifold IV; Locally simply connected*, Annals of Math, to appear, math.AP/0210119.

[7] M. do Carmo and C. K. Peng *Stable complete minimal surfaces in $\mathbb{R}^3$ are planes.*, Bull. Amer. Math. Soc. (N.S.) 1 (1979), no. 6, 903–906.

[8] D. Fischer-Colbrie and R. Schoen, *The structure of complete stable minimal surfaces in 3-manifolds*, Comm. Pure Appl. Math. 33 (1980) 199-211.

[9] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin-New York, 1983.

[10] V. Guillemin and A. Pollack, *Differential Topology*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1974.

[11] A. Hatcher, *Algerbraic Topology*, Cambridge University Press, 2002.

[12] N. Kapouleas, *Complete constant mean curvature surfaces in Euclidean three-space*, Ann. of Math. (2), 131 (1990), no. 2, 239–330.

[13] N. Kapouleas, *Constant mean curvature surfaces in Euclidean spaces*, Proceedings of the International Concress of Mathematicians, Vol. 1, 2 (Zrich, 1994), 481–490, Birkhuser, Basel, 1995.

[14] K. Kenmotsu, *Surfaces with Constant Mean Curvature*, Translations of Mathematical Monographs, v. 221, AMS, 2003.

[15] J. C. Nitsche, *Lecture on Minimal Surfaces, vol. 1, Introduction, Fundamentals, Geometry, and Basic Boundary Value Problems* (English translation), Cambridge University Press, Cambridge, 1989.

[16] R. Osserman: *A Survey of Minimal Surfaces*, Dover Publications, Inc. New York, 1986.

[17] R. Schoen and S.-T. Yau: *Lectures on Differential Geometry*, Conference Proceeding and Lectures Notes in Geometry and Topology, International Press, 1994.

[18] S. Zhang, *Curvature estimates for CMC surfaces in three dimensional manifolds*, preprint.