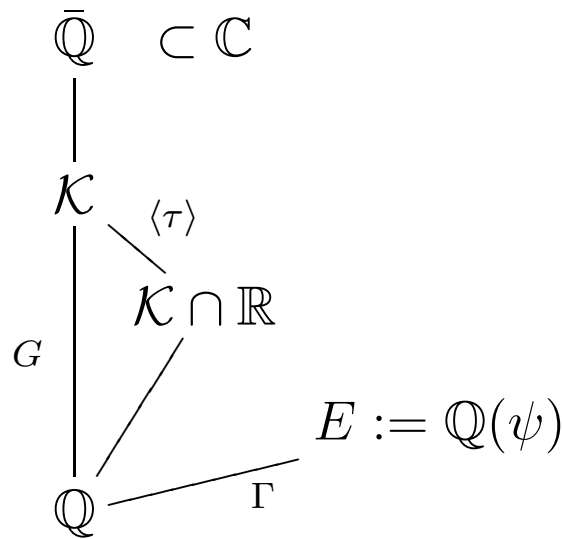


NUMERICAL VERIFICATION OF THE
STARK-CHINBURG CONJECTURE
FOR SOME ICOSAHEDRAL
REPRESENTATIONS

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The Conjecture



$$\rho : G \rightarrow \mathrm{GL}_2(\mathbb{C})$$

odd and irreducible

$$\psi : G \rightarrow \mathbb{C}^\times$$

corresponding character

$$L(s, \rho)$$

corresponding Artin L -function

for $d \in E$, define

$$f_d(s, \rho) := \sum_{\gamma \in \Gamma} d^\gamma L(s, \rho^\gamma)$$

so $f_d(s, \rho) = \sum_{n \geq 1} A_n n^{-s}$ with $A_n \in \mathbb{Q}$ for all $n \geq 1$.

Conjecture (STARK-CHINBURG) *Let $d \in E$ be such that $A_n \in \mathbb{Z}$ for all $n \geq 1$, then there exists a unit $\varepsilon(d) \in \mathcal{K}^+ := \mathcal{K} \cap \mathbb{R}$ such that, for all $\sigma \in G$:*

$$\log \|\varepsilon(d)^{\sigma^{-1}}\| = f'_{d(\psi(\sigma) + \psi(\sigma\tau))}(0).$$

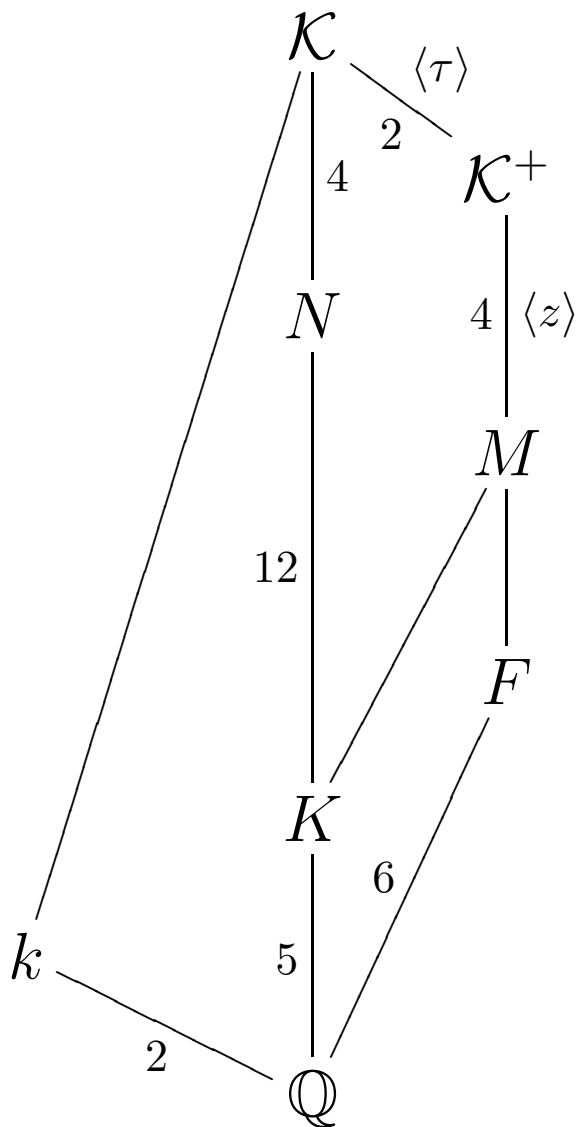
Present Status of the Conjecture

$\bar{\rho} : G \xrightarrow{\rho} \mathrm{GL}_2(\mathbb{C}) \rightarrow \mathrm{PGL}_2(\mathbb{C})$ projective representation

$\mathrm{Im}\bar{\rho} \simeq D_n$ (dihedral)	STARK	proved under some conditions
$\mathrm{Im}\bar{\rho} \simeq A_4$ (tetrahedral)	CHINBURG	numerical examples
$\mathrm{Im}\bar{\rho} \simeq S_4$ (octahedral)	FOGEL	numerical examples

$\mathrm{Im}\bar{\rho} \simeq A_5$ (icosahedral)
14 numerical examples

Constructing examples



$$1 \longrightarrow C_4 \longrightarrow \hat{A}_5 \longrightarrow A_5 \longrightarrow 1 \text{ (non-split)}$$

$$\text{Im } \bar{\rho} \simeq A_5 \simeq \text{PSL}_2(\mathbb{F}_5)$$

$$\text{Im } \rho \simeq \hat{A}_5 \simeq \text{ESL}_2(\mathbb{F}_5) \text{ (determinant } \pm 1)$$

K/\mathbb{Q} a non-real quintic field of type A_5

N/\mathbb{Q} its Galois closure, so $\text{Gal}(N/\mathbb{Q}) \simeq A_5$

\mathcal{K}/\mathbb{Q} obtained by solving $\hat{A}_5 \rightarrow A_5$

$$\psi : G \xrightarrow{\rho} \text{GL}_2(\mathbb{C}) \xrightarrow{\text{Tr}} \mathbb{C}$$

$$\chi : G \xrightarrow{\rho} \text{GL}_2(\mathbb{C}) \xrightarrow{\det} \mathbb{C}^\times \text{ (Gal}(\mathcal{K}/k) = \text{Ker}(\chi))$$

$$E := \mathbb{Q}(\psi) = \mathbb{Q}(\sqrt{5}, i)$$

F/\mathbb{Q} sextic resolvent of K/\mathbb{Q}

$$[M : \mathbb{Q}] = 30 \text{ and } \text{Gal}(\mathcal{K}^+/M) \simeq C_4$$

Some reductions

$E := \mathbb{Q}(\psi) = \mathbb{Q}(\sqrt{5}, i)$, so $\text{Gal}(E/\mathbb{Q}) = \langle \gamma_1, \gamma_2 \rangle$ with

$$\begin{aligned}\gamma_1(i) &= -i, & \gamma_1(\sqrt{5}) &= \sqrt{5}, \\ \gamma_2(i) &= i, & \gamma_2(\sqrt{5}) &= -\sqrt{5}.\end{aligned}$$

Let $d \in E$,

$$f_d(s) := \sum_{\gamma \in \Gamma} d^\gamma L(s, \rho^\gamma) = 2\Re(dL(s, \rho) + d^{\gamma_2} L(s, \rho^{\gamma_2})) = \sum_{n \geq 1} A_n n^{-s}$$

with $A_n = \text{Tr}_{E/\mathbb{Q}}(da_n)$, so $A_n \in \mathbb{Z}$ for all $n \geq 1$ iff

$$d \in \mathcal{D}(E)^{-1} = \frac{1}{2}\mathbb{Z} + \frac{i}{2}\mathbb{Z} + \frac{5 + \sqrt{5}}{20}\mathbb{Z} + i\frac{5 + \sqrt{5}}{20}\mathbb{Z}.$$

Note: $\varepsilon(m d_1) = \varepsilon(d_1)^m$ and $\varepsilon(d_1 + d_2) = \varepsilon(d_1) \times \varepsilon(d_2)$ for $d_1, d_2 \in \mathcal{D}(E)^{-1}$, but also $\varepsilon(i d_1) = \varepsilon(d_1)^z$, so:

Conjecture true for all $d \in \mathcal{D}(E)^{-1} \iff$ Conjecture true for $d = \frac{1}{2}$ and $d = \frac{5 + \sqrt{5}}{20}$.

Computing L -functions

There exists $W \in \mathbb{C}$ with $|W| = 1$ such that, for any $u > 0$:

$$L'(0, \rho) = \frac{W\sqrt{N}}{2\pi} \sum_{n \geq 1} \frac{\overline{a_n}}{n} \exp\left(-\frac{2\pi n}{u\sqrt{N}}\right) + \sum_{n \geq 1} a_n \operatorname{Ei}\left(\frac{2\pi un}{\sqrt{N}}\right)$$

where $N :=$ conductor of ρ and

$$\operatorname{Ei}(x) = \int_x^{+\infty} e^{-t} dt/t$$

exponential integral function.

Method: compute the two sums for two different values of u , deduce W (check $|W| = 1$) and then get $L'(0, \rho)$.

Finding the unit

Conjecture gives $\|\varepsilon^\sigma\|$ for all $\sigma \in G = \text{Gal}(\mathcal{K}/\mathbb{Q})$, but $\varepsilon, \varepsilon^z, \varepsilon^{z^2} = \varepsilon^{-1}$ and $\varepsilon^{z^3} = \varepsilon^{-z}$ are all *real*. Also, since $\text{Gal}(\mathcal{K}^+/M) = \langle z \rangle$:

$$\begin{aligned} P(X) &= (X - \varepsilon)(X - \varepsilon^z)(X - \varepsilon^{z^2})(X - \varepsilon^{z^3}) \\ &= X^4 + aX^3 + bX^2 + aX + 1 \quad \in \mathcal{O}_M[X]. \end{aligned}$$

Need to reconstruct a and b as elements of \mathcal{O}_M with a and b known to a large precision (as real numbers) and with an upper bound for the absolute value of their conjugates over \mathbb{Q} .

Method: Use a special quadratic form, also use $\varepsilon^{1/4}$ or $\varepsilon^{1/2}$ whenever it is possible.

Reconstructing an algebraic integer (2)

Quadratic form on \mathbb{Z}^{31} :

$$Q(v_0, v_1, \dots, v_{30}) = C^2 v_0^2 + (C/\epsilon)^2 \left(v_1 \omega_1^{(1)} + \dots + v_{30} \omega_{30}^{(1)} - v_0 \alpha \right)^2 \\ + \sum_{j=2}^{30} \left(v_1 \omega_1^{(j)} + \dots + v_{30} \omega_{30}^{(j)} \right)^2$$

- If $a = a_1 \omega_1 + \dots + a_{30} \omega_{30}$ is a solution, then $Q(1, a_1, \dots, a_{30}) < 17C^2$.
- If $Q(v_0, v_1, \dots, v_{30}) < 17C^2$ then $|v_0| \leq 4$, and if furthermore $v_0 = \pm 1$, then

$$a := v_0(v_1 \omega_1 + \dots + v_{30} \omega_{30}) \in \mathcal{O}_M$$

satisfies:

$$|a - \alpha| < 4\epsilon, \quad |r_2(a)| < 4C \quad \text{and} \quad |c_j(a)| < 4C \quad (1 \leq j \leq 14).$$

Some further verifications

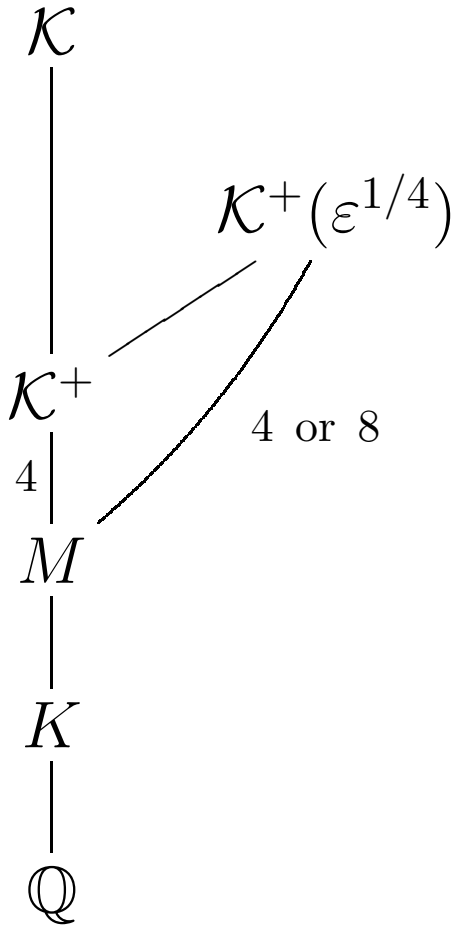
Let $P(X) = \text{Irr}(\tilde{\varepsilon}, M)$ and $Q(X) = \text{Irr}(\tilde{\varepsilon}, \mathbb{Q})$.

- Check roots of $P(X)$ agree with $\exp\left(f'_{d(\psi(\sigma)+\psi(\sigma\tau))}(0)\right)$ for $\sigma \in \langle z \rangle$.
- Check there exists an ordering $\tilde{\varepsilon}_\sigma$ ($\sigma \in G$) on the roots of Q such that

$$\log \|\tilde{\varepsilon}_\sigma\| \simeq f'_{d(\psi(\sigma^{-1})+\psi(\sigma^{-1}\tau))}(0).$$

- J/\mathbb{Q} a degree 24 number field such that $KJ = \mathcal{K}^+$, check that $KJ = \mathbb{Q}(\tilde{\varepsilon})$.
- Galois action?

The abelian condition



$\mathcal{K}^+(\varepsilon^{1/4})/M$ is an abelian
extension
of group C_4 or C_8

The examples

N	d_k	K	power	+twist?
$1948 = 2^2 \cdot 487$	-487	$X^5 - 7X^3 - 17X^2 + 18X + 73$	$1/4$	No
2083	-2083	$X^5 + 8X^3 + 7X^2 + 172X + 53$	$1/4$	No
$2336 = 2^5 \cdot 73$	$-2^2 \cdot 73$	$X^5 + 2X^3 - 4X^2 - 2X + 4$	$1/2$	χ_{-4}
2707	-2707	$X^5 - 2X^4 + 2X^3 - 8X^2 + 21X - 62$	$1/4$	No
$2863 = 7 \cdot 409$	$-7 \cdot 409$	$X^5 - 2X^4 + 6X^3 + X^2 + 14X + 49$	$1/4$	χ_{-7}
$3004 = 2^2 \cdot 751$	-751	$X^5 - 8X^3 + 10X^2 + 160X + 128$	$1/2$	No
3203	-3203	$X^5 - X^4 - X^3 - 9X^2 + 20X - 11$	$1/4$	No
3547	-3547	$X^5 - 8X^3 - 2X^2 + 31X + 74$	$1/4$	No
$3548 = 2^2 \cdot 887$	-887	$X^5 + 10X^3 + 10X^2 + 44X + 56$	$1/2$	No
$3587 = 17 \cdot 211$	$-17 \cdot 211$	$X^5 - 2X^4 - 3X^3 - 13X^2 + 51X - 17$	$1/4$	χ_{17}
$3676 = 2^2 \cdot 919$	-919	$X^5 - 8X^3 + 28X^2 - 40X + 48$	$1/2$	No

$$N = 7 \cdot 409, d = (5 + \sqrt{5})/20$$

$$\begin{aligned}
& + 26X^{119} - 94X^{118} - 13600X^{117} - 281311X^{116} - 3321996X^{115} - 26695600X^{114} - 154147794X^{113} - 633454733X^{112} - 1715520846X^{111} \\
& - 3470848290X^{110} - 30730083146X^{109} - 442488173187X^{108} - 4241975499068X^{107} - 29683203312874X^{106} - 160018034060140X^{105} \\
& - 663689188701923X^{104} - 2021804070374122X^{103} - 4006528158838154X^{102} - 4884414082098948X^{101} - 36581684427924657X^{100} \\
& - 487528951945104522X^{99} - 3783005384356929126X^{98} - 20403327017395397024X^{97} - 84955151957338226945X^{96} - 294293366619039740386X^{95} \\
& - 913545334089156134170X^{94} - 2697514791586967527902X^{93} - 7663680434531394529531X^{92} - 20456291234678745439466X^{91} \\
& - 51289870206760479995338X^{90} - 127082471589904301709030X^{89} - 325889924154119365692755X^{88} - 836465690752396946216978X^{87} \\
& - 1996086627778243950531172X^{86} - 4292965781360113755325812X^{85} - 8612234037145139046881703X^{84} - 17205248221768863457571082X^{83} \\
& - 34980994234137834004988972X^{82} - 69290016038721990395949600X^{81} - 128003082848154913592158939X^{80} - 220610572593979107867316098X^{79} \\
& - 364225668247380528514085586X^{78} - 578852895069490862382330272X^{77} - 850511718430215025448509179X^{76} - 1081353097405617020513141512X^{75} \\
& - 096185494282559030925178618X^{74} - 742769471405376488763892122X^{73} - 15033934417574738736045691X^{72} + 914406880475444830539894282X^{71} \\
& + 1775424928237598246145749032X^{70} + 2320289738861916855748839350X^{69} + 2425873616614254748075556939X^{68} \\
& + 2187960822311042822094511360X^{67} + 1983824193814413999886771962X^{66} + 2372728685456958572223960652X^{65} \\
& + 3760223467003604255883124473X^{64} + 6039722171959017044736565850X^{63} + 8564829773982177534419666602X^{62} \\
& + 10499011099495497101938960668X^{61} + 11217034388314946629995466683X^{60} + 10499011099495497101938960668X^{59} \\
& + 8564829773982177534419666602X^{58} + 6039722171959017044736565850X^{57} + 3760223467003604255883124473X^{56} \\
& + 2372728685456958572223960652X^{55} + 1983824193814413999886771962X^{54} + 2187960822311042822094511360X^{53} \\
& - 25873616614254748075556939X^{52} + 2320289738861916855748839350X^{51} + 1775424928237598246145749032X^{50} + 914406880475444830539894282X^{49} \\
& - 5033934417574738736045691X^{48} - 742769471405376488763892122X^{47} - 1096185494282559030925178618X^{46} - 1081353097405617020513141512X^{45} \\
& - 50511718430215025448509179X^{44} - 578852895069490862382330272X^{43} - 364225668247380528514085586X^{42} - 220610572593979107867316098X^{41} \\
& - 128003082848154913592158939X^{40} - 69290016038721990395949600X^{39} - 34980994234137834004988972X^{38} - 17205248221768863457571082X^{37} \\
& - 8612234037145139046881703X^{36} - 4292965781360113755325812X^{35} - 1996086627778243950531172X^{34} - 836465690752396946216978X^{33} \\
& - 325889924154119365692755X^{32} - 127082471589904301709030X^{31} - 51289870206760479995338X^{30} - 20456291234678745439466X^{29} \\
& - 7663680434531394529531X^{28} - 2697514791586967527902X^{27} - 913545334089156134170X^{26} - 294293366619039740386X^{25} \\
& - 84955151957338226945X^{24} - 20403327017395397024X^{23} - 3783005384356929126X^{22} - 487528951945104522X^{21} \\
& - 36581684427924657X^{20} - 4884414082098948X^{19} - 4006528158838154X^{18} - 2021804070374122X^{17} - 663689188701923X^{16} \\
& - 160018034060140X^{15} - 29683203312874X^{14} - 4241975499068X^{13} - 442488173187X^{12} - 30730083146X^{11} - 3470848290X^{10} \\
& - 1715520846X^9 - 633454733X^8 - 154147794X^7 - 26695600X^6 - 3321996X^5 - 281311X^4 - 13600X^3 - 94X^2 + 26X + 1
\end{aligned}$$