Cup products and local embeddings of p-units

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joint with W. McCallum

Notation: $n \geq 1$
K number field with $\mu_n \subset K$
S finite set of primes, $\{v \mid n\infty\} \subseteq S$
K_S max. extension of K unramified outside S
$G_{K,S} = \mathrm{Gal}(K_S/K)$
$\mathcal{O}_{K,S} = S$-integers of K
$\mathrm{Cl}_{K,S} = S$-class group of K
$I_{K,S} = S$-ideal group of K
$\mathrm{Br}_S(K) = \ker(\oplus_{v \in S} \mathrm{Br}(K_v) \to \mathbf{Q}/\mathbf{Z})$

Cup product:

$$H^1(G_{K,S}, \mu_n) \otimes H^1(G_{K,S}, \mu_n) \xrightarrow{\cup} H^2(G_{K,S}, \mu_n^{\otimes 2})$$

The cohomology groups:
The Kummer sequence

$$1 \to \mu_n \to \mathcal{O}_{K_S,S}^\times \to \mathcal{O}_{K_S,S}^\times \to 1$$

yields

$$1 \to \mathcal{O}_{K,S}^\times / \mathcal{O}_{K,S}^{\times n} \to H^1(G_{K,S}, \mu_n) \to \mathrm{Cl}_{K,S}[n] \to 1$$
$$1 \to \mathrm{Cl}_{K,S}/n \to H^2(G_{K,S}, \mu_n) \to \mathrm{Br}_S(K)[n] \to 1$$

Rephrasing of interpretation of H^1:

$$H^1(G_{K,S},\mu_n) \cong D_K/K^{\times n},$$
$$D_K = \{a \in K^\times \colon a\mathcal{O}_{K,S} \in nI_{K,S}\}$$

We define a pairing:

$$(\ ,\ )_S = (\ ,\ )_{n,K,S} \colon D_K \times D_K \xrightarrow{\cup} H^2(G_{K,S},\mu_n^{\otimes 2})$$

The projection of $(\ ,\ )_S$ to $Br_S(K)[n] \otimes \mu_n$ is the sum of norm residue symbols at $v \in S$. We consider the case that this projection is trivial.

A formula for the pairing:

Theorem 1. *Let $a, b \in D_K$ satisfy $(a,b)_{n,K_v} = 1$ for all $v \in S$. Write $b\mathcal{O}_{K,S} = \mathfrak{b}^n$, $\alpha^n = a$, $L = K(\alpha)$, $m = [L : K]$, and $b = N_{L/K}\gamma$ for $\gamma \in L^\times$. We may write*

$$\gamma\mathcal{O}_{L,S} = \mathfrak{c}^{1-\sigma}\mathfrak{b}^{n/m}$$

for some ideal $\mathfrak{c}$ and element $\sigma \in$ Gal(L/K). Then

$$(a,b)_S = N_{L/K}\mathfrak{c} \cdot \mathfrak{b}^{n(m-1)/2} \otimes \alpha^{\sigma-1}.$$

Remarks.

1. The contribution from $\mathfrak{b}^{n(m-1)/2}$ has order dividing 2 and is trivial if m is odd.

2. Taking σ to be a generator of Gal(L/K), the ideal in the theorem is the same as $\prod_{i=1}^{m-1} \sigma^i \gamma^i \cdot \mathcal{O}_{L,S}$ modulo $mI_{L,S}$.

Corollary 1. *If* $b \in N_{L/K}\mathcal{O}_{L,S}^\times$ *then* $(a,b)_S = 0$.

Corollary 2. *If* $a, 1-a \in \mathcal{O}_{K,S}^\times$ *then* $(a, 1-a)_S = 0$.

Relationship with K-theory.

$$K_2^M(\mathcal{O}_{K,S}) = \frac{\mathcal{O}_{K,S}^\times \otimes \mathcal{O}_{K,S}^\times}{\langle x \otimes (1-x) \mid x, 1-x \in \mathcal{O}_{K,S}^\times \rangle}$$

$$\begin{array}{ccccccccc} & & K_2^M(\mathcal{O}_{K,S}) & \longrightarrow & K_2^M(K) & & & & \\ & & \downarrow & & \| & & & & \\ 0 & \longrightarrow & K_2(\mathcal{O}_{K,S}) & \longrightarrow & K_2(K) & \longrightarrow & \oplus_{v \notin S} k_v^\times & \longrightarrow & 0 \end{array}$$

Theorem 2 (Tate,Soulé). *There is a commutative diagram*

$$\begin{array}{ccc} K_2^M(\mathcal{O}_{K,S})/n & \longrightarrow & K_2(\mathcal{O}_{K,S})/n \\ & \searrow & \downarrow \\ & & H^2(G_{K,S}, \mu_n^{\otimes 2}) \end{array}$$

where the horizontal map is the natural map, the diagonal map is induced by $(\ ,\ )_S$, and the vertical map is an isomorphism given by a Chern class map.

Our focus: $n = p$, an odd prime
$K = \mathbf{Q}(\zeta_p)$, $S = \{(1 - \zeta_p)\}$

Remarks:
1. $\mathrm{Cl}_{K,S} = \mathrm{Cl}_K$.
2. $\mathrm{Br}_S(K) = 0$, so

$$H^2(G_{K,S}, \mu_p^{\otimes 2}) \cong \mathrm{Cl}_K \otimes \mu_p.$$

More notation: $\Delta = \mathrm{Gal}(K/\mathbf{Q})$
$\omega\colon G_{\mathbf{Q}} \to \mathbf{Z}_p$, the Teichmüller character
Idempotents:

$$\epsilon_i = \frac{1}{p-1} \sum_{\delta \in \Delta} \omega(\delta)^{-i} \delta$$

Assumption (cyclicity conjecture):
$(\mathrm{Cl}_K \otimes \mathbf{Z}_p)^{\epsilon_i}$ is cyclic for (odd) i.

This conjecture was a question of Iwasawa's.
It is a consequence of Vandiver's conjecture.

For $r \geq 2$ even, $(\mathrm{Cl}_K \otimes \mathbf{Z}_p)^{\epsilon_{1-r}} \neq 0$ if and only if $p \mid B_r$.

Assume $p \mid B_r$. We call (p, r) an irregular pair. Choose an isomorphism

$$\iota\colon (\mathrm{Cl}_K \otimes \mu_p)^{\epsilon_{2-r}} \xrightarrow{\sim} \mathbf{Z}/p\mathbf{Z}(2-r),$$

defined by $\mathfrak{a}_r \otimes \zeta \mapsto 1$.

Galois equivariant pairing:

$$\langle\ ,\ \rangle_r = \iota \circ \epsilon_{2-r} \circ (\ ,\ )_S\colon D_K \times D_K \to \mathbf{Z}/p\mathbf{Z}(2-r)$$

Pairing with ζ:
Eisenspace considerations yield $\langle \zeta, \mathcal{O}_{K,S}^{\times} \rangle_r = 0$. Under the cyclicity conjecture, the inverse limit A_∞ of p-parts of class groups up the cyclotomic $\mathbf{Z}_p$-tower has the form

$$A_\infty^{\epsilon_{1-r}} \cong \mathbf{Z}_p[[T]]/(f_r(T)).$$

Let $a_r \mathcal{O}_{K,S} = \mathfrak{a}_r^{f_r(0)/p}$.

Proposition 1. *$\langle \zeta, a_r \rangle_r = -f_r'(0) \pmod{p}$.*

Remark: $f_r'(0) \not\equiv 0 \pmod{p}$ if and only if the λ-invariant of $A_\infty^{\epsilon_{1-r}}$ is 1.
Buhler-C.-E.-M.-S. have verified $\lambda = 1$ for $p < 12{,}000{,}000$.

More notation:

$\mathcal{C}$ cyclotomic p-units

For odd i, choose $\eta_i \in \mathcal{C}^+$,

$$\eta_i \equiv (1-\zeta)^{\epsilon_{1-i}} \pmod{\mathcal{C}^p}.$$

Note: $\langle \eta_i, \eta_j \rangle_r = 0$ unless $i + j \equiv r \pmod{p-1}$.

Set $e_{i,r} = \langle \eta_i, \eta_{r-i} \rangle_r$.

Technical issue:

Let

$$q_i = |(\mathrm{Cl}_K \otimes \mathbf{Z}_p)^{\epsilon_{1-i}}|,$$

and take $\alpha_i \in (\mathcal{O}_{K,S}^\times)^+$ with $\alpha_i^{q_i} \equiv \eta_i \pmod{\mathcal{C}^p}$.

Note: $q_i = 1$ if Vandiver holds at p.

If $q_i > 1$, then $e_{i,r} = 0$, and one should consider $\langle \alpha_i, \alpha_{r-i} \rangle_r$.

To compute possible values of the $e_{i,r}$, we impose relations.

Some relations in Milnor K-theory:

a. $\zeta^a(\zeta^b - 1) + (\zeta^a - 1) = \zeta^{a+b} - 1$

b. $(\zeta^{a+b+c} - 1)(\zeta^a - 1) + \zeta^a(\zeta^b - 1)(\zeta^c - 1)$
$= (\zeta^{a+b} - 1)(\zeta^{a+c} - 1)$

c. $(\zeta^{a+b} - 1)(\zeta^{2a} - 1) + \zeta^a(\zeta^a - 1)(\zeta^b - 1)(\zeta^{a+b} - 1)$
$= (\zeta^a - 1)(\zeta^{2(a+b)-1})$

d. $\zeta^b(\zeta^{3a} - 1)(\zeta^{a+b} - 1)$
$+ (\zeta^a - 1)(\zeta^b - 1)(\zeta^{a+b} - 1)(\zeta^{2a+b} - 1)$
$= (\zeta^a - 1)(\zeta^{3(a+b)} - 1)$

e. $\zeta^a(\zeta^{4b} - 1)(\zeta^{a+b} - 1)(\zeta^{a+b} - 1)(\zeta^{a+2b} - 1)$
$+ (\zeta^a - 1)(\zeta^b - 1)(\zeta^{a+b} - 1)(\zeta^{a+2b} - 1)(\zeta^{a+3b} - 1)$
$= (\zeta^{2a+4b} - 1)(\zeta^{2a+2b} - 1)(\zeta^b - 1)$

f. $(\zeta^a - 1)(\zeta^{2(a+b)} - 1)(\zeta^{a+c} - 1)(\zeta^c - 1)$
$+ \zeta^a(\zeta^a - 1)(\zeta^b - 1)(\zeta^{2c} - 1)(\zeta^{a+b} - 1)$
$= (\zeta^{a+b+c} - 1)(\zeta^{2a} - 1)(\zeta^c - 1)(\zeta^{a+b} - 1)$

We consider only the relations (c) with b odd, $1 \leq b \leq p-2$ and $a = 1$.
Note that these may be rewritten:

$$\frac{1-\zeta^{b+1}}{1+\zeta} + \zeta\frac{1+\zeta^{b}}{1+\zeta} = 1.$$

Note: If $\alpha, \beta \in \mathcal{C}$ then

$$\langle \alpha, \beta \rangle_r = \sum_{\substack{i=1 \\ i \text{ odd}}}^{p-2} \langle \alpha^{\epsilon_i}, \beta^{\epsilon_{r-i}} \rangle_r,$$

and for c with $p \nmid c$,

$$(1-\zeta^{c})^{\epsilon_{1-i}} \equiv \eta_i^{c^{i-1}} \pmod{\mathcal{C}^p}.$$

From these and our relations, we obtain

$$\sum_{\substack{i=1 \\ i \text{ odd}}}^{p-2} (1+(b+1)^{p-i}-2^{p-i})(1-2^{p-r+i})(1-b^{p-r+i})x_i = 0 \quad (1)$$

for $x_i = e_{i,r}$ (and odd b as before).

Theorem 3. *For all irregular pairs* (p, r) *with* $p < 10,000$*, there exists a nontrivial, Galois equivariant, skew-symmetric pairing*

$$\langle\ ,\ \rangle \colon \mathcal{C} \times \mathcal{C} \to \mathbf{Z}/p\mathbf{Z}(2-r)$$

satisfying (1) with $x_i = \langle \eta_i, \eta_{r-i} \rangle$*, and it is unique with these properties up to a scalar multiple.*

Corollary 3. *For all irregular pairs* (p, r) *with* $p < 10,000$*, one has*

$$|(K_2^M(\mathcal{O}_{K,S})/p)^{\epsilon_{2-r}}| \leq p.$$

In fact, we computed:

Proposition 2. *For all irregular pairs* (p, r) *with* $p < 4,000$*, one has*

$$|(K_2^M(\mathcal{O}_{K,S}) \otimes \mathbf{Z}_p)^{\epsilon_{2-r}}| \leq p.$$

A surjectivity conjecture:

Conjecture. *For any odd p satisfying Vandiver's conjecture, the map*

$$K_2^M(\mathcal{O}_{K,S}) \otimes \mathbf{Z}_p \to K_2(\mathcal{O}_{K,S}) \otimes \mathbf{Z}_p$$

is surjective.

Remarks:

1. If p violates Vandiver's conjecture, but satisfies the cyclicity conjecture, this conjecture still probably holds. If one of the even eigenspaces of the p-class group has p-rank ≥ 2, then one would need to take minus parts.
2. We make no conjecture about injectivity. For instance, the relations we have are not always sufficient to show that eigenspaces of K_2^M are 0 for regular pairs (p, r).

Zeroes of the pairing:

Remark: The pairing is not nondegenerate. In fact, $e_{p-r,r} = 0$ since, for $\gamma^p = \eta_{p-r}$,

$$(N_{K(\gamma)/K} \mathrm{Cl}_{K(\gamma),S} \otimes \mathbf{Z}_p)^{\epsilon_{1-r}} = 0.$$

Table of pairings:

p = 37, r = 32
(1 26 0 36 1 35 31 34 3 6 2 36 1 0 11 36 11 26)

p = 59, r = 44
(1 45 21 30 14 35 5 0 48 57 7 52 2 11 0 54 24 45 29
38 14 58 27 32 15 0 44 27 32)

p = 67, r = 58
(1 45 38 56 0 47 62 9 29 15 65 26 45 57 0 10 22 41 2
52 38 58 5 20 0 11 29 22 66 2 24 43 65)

p = 101, r = 68
(1 56 40 96 26 63 0 61 81 71 35 92 73 64 6 88 0 0 13
95 37 28 9 66 30 20 40 0 38 75 5 61 45 100 17 17 12
66 72 53 86 31 70 15 48 29 35 89 84 84)

p = 103, r = 24
(1 70 17 22 77 25 78 26 81 86 33 102 18 4 26 92 77
54 88 90 23 26 57 0 11 86 70 85 85 97 57 0 46 6 18
18 33 17 92 0 46 77 80 13 15 49 26 11 77 99 85)

p = 131, r = 22
(1 35 74 129 81 0 50 2 57 96 130 0 38 8 81 67 83 64
3 127 107 0 34 69 23 105 34 64 100 105 70 73 37 13
118 114 124 36 95 7 17 13 118 94 58 61 26 31 67 97
26 108 62 97 0 24 4 128 67 48 64 50 123 93 0)

Relationship with K-theory II:

We remark that α_i (with $\alpha_i^{q_i} = \eta_i$) has nonzero image $\bar{\alpha}_i$ in

$H^1(G_{K,S}, \mathbf{Z}/p\mathbf{Z}(i))^{\Delta} \cong (H^1(G_{K,S}, \mu_p)(i-1))^{\Delta}$.

We have the restriction map

$$H^1_{et}(\mathbf{Z}[1/p], \mathbf{Z}/p\mathbf{Z}(i)) \xrightarrow{\sim} H^1(G_{K,S}, \mathbf{Z}/p\mathbf{Z}(i))^{\Delta},$$

and $\bar{\alpha}_i$ generates the image of the composition of the natural map

$$H^1_{cts}(\mathbf{Z}[1/p], \mathbf{Z}_p(i)) \to H^1_{et}(\mathbf{Z}[1/p], \mathbf{Z}/p\mathbf{Z}(i)).$$

with restriction.

By results of Soulé and Dwyer-Friedlander, we have surjections

$$\mathrm{ch}_{i,k} \colon K_{2i-k}(\mathbf{Z}) \otimes \mathbf{Z}_p \to H^k(\mathbf{Z}[1/p], \mathbf{Z}_p(i)) \quad (2)$$

for $k = 1, 2$ and $i \geq 1$.

Conjecture (Quillen-Lichtenbaum). The maps $\mathrm{ch}_{i,k}$ are isomorphisms.

The Quillen-Lichtenbaum conjecture has reportedly been proven by Voevodsky and Rost.

We have a commutative diagram:

$$\begin{array}{ccc} K_{2i-1}(\mathbf{Z}) \otimes K_{2j-1}(\mathbf{Z}) & \longrightarrow & K_{2(i+j)-2}(\mathbf{Z}) \otimes \mathbf{Z}_p \\ \downarrow \mathrm{ch}_{i,1} \otimes \mathrm{ch}_{j,1} & & \downarrow \mathrm{ch}_{i+j,2} \\ H^1(\mathbf{Z}[\frac{1}{p}], \mathbf{Z}_p(i)) \otimes H^1(\mathbf{Z}[\frac{1}{p}], \mathbf{Z}_p(j)) & \longrightarrow & H^2(\mathbf{Z}[\frac{1}{p}], \mathbf{Z}_p(i+j)) \end{array}$$

where the top horizontal map is the product.

Let $b_i \in K_{2i-1}(\mathbf{Z})$ have image $\bar{\alpha}_i$.

Theorem 4. *If* $b_i \cdot b_{r-i} \equiv 0 \pmod{p}$ *then*

$$\langle \alpha_i, \alpha_{r-i} \rangle_r = 0,$$

and the Quillen-Lichtenbaum conjecture implies the converse.

Remark. The pairing is more precisely analogous to products in $K_\cdot(\mathbf{Z}; \mathbf{Z}/p)$.

An approach to nontriviality:

Let $L = K(\eta_{p-r}^{1/p})$, an unramified cyclic extension K of degree p.

Consider the commutative diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathsf{Cl}_{L,S}/p\mathsf{Cl}_{L,S} & \longrightarrow & H^2(G_{L,S},\mu_p) & \longrightarrow & \bigoplus^0_{\mathfrak{p}|p} H^2(L_{\mathfrak{p}},\mu_p) & \longrightarrow 0 \\
& \downarrow{\scriptstyle N_{L/K}} & & \downarrow{\scriptstyle \mathsf{Cor}} & & \downarrow{\scriptstyle f} & \\
0 \rightarrow & N_{L/K}\mathsf{Cl}_{L,S}/p\mathsf{Cl}_{K,S} & \rightarrow & \mathsf{Cl}_{K,S}/p\mathsf{Cl}_{K,S} & \xrightarrow{\pi} & \mathsf{Cl}_{K,S}/N_{L/K}\mathsf{Cl}_{L,S} & \rightarrow 0
\end{array}$$

Let σ generate $\mathsf{Gal}(L/K)$.
For any $z \in L^{\times}$, let $D = \sum_{k=1}^{p-1} k\sigma^k$.
Fix a prime $\mathfrak{p}_0$ of L above p.

Theorem 5. *Suppose that* $b = N_{L/K}\beta$ *for some* $\beta \in \mathcal{O}_{L,S}^{\times}$. *Then*

$$(\pi \otimes \mathrm{id})(a,b)_S = \mathfrak{c} \otimes (a, \beta^D)_{p,L_{\mathfrak{p}_0}}$$

for some ideal class $\mathfrak{c}$ *with image generating* $\mathsf{Cl}_{K,S}/N_{L/K}\mathsf{Cl}_{L,S}$.

Remark. If $|\mathrm{Cl}_K \otimes \mathbf{Z}_p| = p$, then $N_{L/K}\mathcal{O}_{L,S}^{\times} = \mathcal{O}_{K,S}^{\times}$, and π is an isomorphism.

We let Δ_0 denote the subgroup of $\mathrm{Gal}(L/K)$ fixing $\mathfrak{p}_0$.

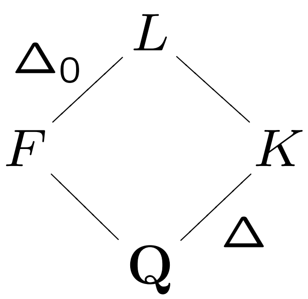
Proposition 3. *Let $3 \leq i \leq p-2$ be odd such that $p \nmid B_{p-i}$. Assume $\eta_{r-i} = N_{L/K}\beta$, with β chosen to have image contained in the ϵ_{p-r+i}-eigenspace of $\mathcal{O}_{L,S}^{\times}/\mathcal{O}_{L,S}^{\times p}$ under Δ_0. Then $e_{i,r} \neq 0$ if and only if $\beta^D \notin L_{\mathfrak{p}_0}^{\times p}$.*

Question: Does there exist $\beta \in \mathcal{O}_{L,S}^{\times}$ with $\beta^D \notin L_{\mathfrak{p}_0}^{\times p}$?

For $p = 37$, this is computationally verifiable.

Theorem 6. *The pairing $\langle\ ,\ \rangle_{32}$ for $p = 37$ is nontrivial. Thus, the surjectivity conjecture is true for $p = 37$.*

Idea of proof.
With the help of W. Stein and C. Fieker, we determined the p-unit group of the fixed field F of Δ_0 as follows.



The field F is generated by the trace x of a 37th root of η_5, and we found a minimal polynomial for this element using CRT. Various Magma routines then took 5 days on a 2GHz processor to produce an optimal representation of the integer ring of F, from which its p-unit group was computable. One of the p-units β had norm p. By computing the various embeddings of x in the $L_{\mathfrak{p}}$, we were able to verify the condition of the proposition for β^D.

The minimal polynomial of x:

$x^{37} - 6483584x^{34}$
$- 118234637824x^{33}$
$- 123335506765824x^{32}$
$- 7894900273815552x^{31}$
$- 255848961417810247 68x^{30}$
$- 196127866668139920097 28x^{29}$
$- 2221784070205669762924544x^{28}$
$- 336280142496662926329034 83392x^{27}$
$- 4805711697609190244214712041472x^{26}$
$- 22490026154268639920058485115453 44x^{25}$
$- 1309975549653920931146883229082556 8256x^{24}$
$- 31717874363193835017038136769405979197 44x^{23}$
$+ 476259323830076662111107898811789814530048x^{22}$
$- 1396232608839552259966984463923520026947092480x^{21}$
$- 331493134727514939719441018060252656606965137408x^{20}$
$- 80268638062435074559599184759300711777564488630272x^{19}$
$- 8720575656721364925618242048178120979952828721680875$
$52x^{18}$
$+ 1772659418875854490177280483057352783210247369401565$
$184x^{17}$
$+ 3724422223633487548164125253859655282863175862268729$
$9108864x^{16}$
$- 2065140478547750146788189515335798341552634994293825$
$6921329664x^{15}$
$+ 3118354412560871576377464195599807837437444537079124$
$122814696652 8x^{14}$
$+ 2854705449484624416795330612386811215415869973011706$
$932441160613888x^{13}$
$- 1855731458356048530821147730152877548185437344079899$
$163926475684446208 0x^{12}$
$+ 3087405021478910646130093242279350919332930043815268$
$7471639992995434987 52x^{11}$
$- 8448611691348801851628818131891130395295947814515408$
$1673626372646913232 0768x^{10}$

$-1816305918878969636874702964809165551139833636304814$
$6870204995155207780931993 6x^9$
$+ 4849659113957649708716655446098404792782070205898868$
$4410968850588336124204993740 8x^8$
$+ 5511404977678219958262233454095746148362443395726320$
$712307332651607429387602886656 0x^7$
$- 1068505898256327898948966128873297210949111791350824$
$101005198703230324211961842945228 80x^6$
$+ 1225313897986066030513017243242734500242304084259199$
$9699890814819111998281760100895948 8x^5$
$+ 1982469259694895457314935195126430029297012127795195$
$50139655256616546409226918529127258521 6x^4$
$- 6606779528917029897795831254093541405143192723743455$
$08868647000437128182296246506433818918912x^3$
$- 2649881947577745980599970091092292918947828671888834$
$3376506793487843882690105131796149356095078 4x^2$
$+ 6728762060805419616052269030627338051927895313522501$
$37517709918865110475053588494379165834704584704x$
$-26229302920145682793735730674797865320906253597999312$
$3331515899375253384527718903616471614294706880512$

A better polynomial for F:

$$y^{37} + 4y^{36} + 12y^{35} + 36y^{34} - 336y^{33} - 268y^{32} - 3912y^{31} - 7555y^{30} + 60363y^{29} - 254771y^{28} + 1584299y^{27} - 4912687y^{26} + 17776688y^{25} - 51189497y^{24} + 135760742y^{23} - 339845565y^{22} + 729194231y^{21} - 1823351247y^{20} + 2954679204y^{19} - 7136330744y^{18} + 14870105096y^{17} - 19798475744y^{16} + 63485328194y^{15} - 69489469832y^{14} + 240906930339y^{13} - 130150428853y^{12} + 883058481925y^{11} - 525666202335y^{10} + 1336924708802y^{9} - 2790390347185y^{8} + 2312809893723y^{7} - 3005373888911y^{6} + 6491297663291y^{5} - 2826510585529y^{4} + 4902736951337y^{3} - 6453741855514y^{2} + 3673618997547y - 1546779831802$$

The embedding of β^D:

$$-445 + 13\cdot 37t - 3t^{31} - 9t^{32} + 18t^{33} + 14t^{34} + 2t^{35} + O(t^{38}),\ t = \zeta - 1$$

A few consequences for $p = 37$:

The group $K_2^M(\mathcal{O}_{K,S}) \otimes \mathbf{Z}_{37}$ has order 37.

Under the Quillen-Lictenbaum conjecture, the product maps

$K_{2i-1}(\mathbf{Z}) \otimes K_{63+72k-2i}(\mathbf{Z}) \to K_{62+72k}(\mathbf{Z}) \otimes \mathbf{Z}_{37}$

are surjective for a given odd i and any k if $i \not\equiv 5, 27 \bmod 36$ and zero otherwise. (Without Quillen-Lichtenbaum, we have non-triviality and nonsurjectivity in the two respective cases.)

Theorem 7. *Let M/K be a cyclic extension of degree 37 that is unramified outside 37. Then $|\mathrm{Cl}_{M,S} \otimes \mathbf{Z}_{37}| = 37$ if and only if*

$$M \not\subset \mathbf{Q}(\zeta_{37^2}, \eta_5^{1/37}, \eta_{27}^{1/37}).$$

Furthermore, only $M = K(\eta_5^{1/37})$ has trivial 37-class number.