A Refinement of Stark's Conjecture over a Complex Cubic Number Field

A report on the Ph.D. thesis of **Tian Ren** - a student at Rutgers Newark.

Let F be a complex cubic number field, $F^{(1)} \subset \mathbb{R}$, $F = F^{(2)} \subset \mathbb{C}$, $F^{(3)} = \bar{F} \subset \mathbb{C}$, $\mathfrak{f}$ an integral ideal in $\mathfrak{O}_F$, η be a fundamental unit of F such that $\eta^{(1)} > 1$, $Cl_F(\mathfrak{f})$ the usual ray class group modulo $\mathfrak{f}$ in the narrow sense. That group can be viewed as the Galois group $G = Gal(K/F)$, where K is the corresponding ray class field modulo $\mathfrak{f}$. Let $C \in Cl_F(\mathfrak{f})$ correspond to $\sigma \in G$, then the associated partial zeta function is defined by

$$\zeta_F(s, \sigma) = \zeta_F(s, C) = \sum_{\mathfrak{a} \in C} N(\mathfrak{a})^{-s},\ Re(s) > 1,$$

where $\mathfrak{a}$ runs through all integral ideals in the ray class C. In this situation, the conjecture of Stark predicts the existence of a unit $\varepsilon \in \mathfrak{O}_K^*$ in the class field K such that

$$\zeta_F'(0, \sigma) = -\frac{1}{w_K} \log |\sigma(\varepsilon)|^2,$$

where w_K is the number of roots of unity in K.

The problem which I want to report on today is motivated by the following question: To what extent can we get inside the absolute value sign and obtain a useful formula for $\sigma(\varepsilon)$ instead of $|\sigma(\varepsilon)|$? This question is already solved for the case of a totally real base field F and for the case of a complex quadratic base field F. In fact, the known proof on Stark's conjecture in the complex quadratic case relies on a careful study of a formula for the Stark unit ε (which goes back in principle to Kronecker).

By definition of a narrow ray class, we have

$$\begin{aligned}\zeta_F(C,s) &= N(\mathfrak{b})^{-s}\sum_{\mathfrak{a}\sim\mathfrak{b}(\mathfrak{f})} N(\mathfrak{a}\mathfrak{b}^{-s})^{-s} \\ &= N(\mathfrak{b})^{-s}\sum_{\alpha\in 1+\mathfrak{f}\mathfrak{b}^{-1}/U_{\mathfrak{f}}^{+},\alpha>>0} N(\alpha)^{-s},\end{aligned}$$

where $\mathfrak{b}$ is a fixed integral representative of the ray class C, and $U_{\mathfrak{f}}^{+} = \{\omega \in \mathfrak{O}_F^{*} | \omega \equiv 1(\mathfrak{f}), \omega^{(1)} > 0\}$.

In calculating the derivative at $s = 0$, we follow the approach of Shintani. The key idea is a choice of a special fundamental domain for the action of the unit group of F on $\mathbb{R}\times\mathbb{C}$. That fundamental domain is given by a finite collection of open simplicial cones $C_k(v_{k1},\cdots,v_{kr(k)})$ ($k \in S$ is a finite set of indicies,and $r(k) =$ the dimension of the cone C) with generators $v_{k1},\cdots,v_{kr(k)}$. Without loss of generality, we can assume that the generators $v_{k1},\cdots,v_{kr(k)}$ are part of a $\mathbb{Z}$-basis of the fractional ideal $\mathfrak{f}\mathfrak{b}^{-1}$. Under this assumption each

$\alpha \in C_k \cap (1 + \mathfrak{f}\mathfrak{b}^{-1})$ admits the unique expression:

$$\alpha = \sum_{i=1}^{r(k)} (x_i + m_i) v_{ki},$$

where $m_i = 0, 1, 2, \cdots$, and $0 \leqslant x_i < 1$ not all zero. In this way, we can write for $Re(s) > 1$,

$$\zeta_F(s, C) = N(\mathfrak{b})^{-s} \sum_{k \in S} \sum_{m \in \mathbb{Z}_{\geqslant 0}^{r(k)}} N\left(\sum_{i=1}^{r(k)} (x_i + m_i) v_{ki} \right)^{-s}.$$

Our next goal is to calculate the derivative at $s = 0$ of

$$\sum_{m \in \mathbb{Z}_{\geqslant 0}^{r(k)}} N\left(\sum_{i=1}^{r(k)} (x_i + m_i) v_{ki} \right)^{-s}, \quad Re(s) > 1,$$

in terms of multiple Gamma function (due to Barnes). We mention in passing, that the value at $s = 0$ of the above zeta function associated to a cone is a rational number. This remark will be important in a moment.

For $n = 1, 2, 3, \cdots$, we define the *Barnes gamma function* via

$$\log \gamma_n(z, \omega) = \frac{1}{2\pi i} \int_{I_{\varepsilon(+\infty)}} \frac{e^{-zt}(\log t \ + \gamma - \pi i)}{\prod_{i=1}^{n} (1 - e^{-\omega_i t})} \frac{dt}{t},$$

where $I_{\varepsilon(+\infty)}$ is a path starting from $+\infty$ and going to $+\varepsilon$ along the real line, then circling the origin in the anti-clockwise direction on a circle of radius ε and returning to $+\infty$. The integral is well-defined for n-tuple of complex number ω_i with non-zero real part $Re(\omega_i) \neq 0$, and a complex number z with positive real part.

Example: if $Re(\omega) > 0$, then

$$\log \gamma_1(z, \omega) = \log \frac{\Gamma(z/\omega)}{\sqrt{2\pi}} + \left(\frac{z}{\omega} - \frac{1}{2} \right) \log \omega.$$

In general, $\log \gamma_n(z, \omega)$ is equal to

$$\frac{d}{ds} \sum_{m \in \mathbb{Z}^n_{\geqslant 0}} \frac{1}{(z + m_1\omega_1 + \cdots + m_n\omega_n)^s} \bigg|_{s=0}$$

up to an elementary expression (Bernoulli polynomials).

Going back to a cone C, we associate to it a complex matrix V of size $r \times 3$, the row vectors are given by the 3 embeddings of the generators v_j of C into $\mathbb{R} \times \mathbb{C} \times \mathbb{C}$. Before stating the main result, we need to define one more function associated to the matrix V. Depending on the dimension r of the cone C $(r = 1, 2, 3)$, we define

$$\begin{aligned}
\mathfrak{B}_1(x, V, j) &= 0 \\
\mathfrak{B}_2(x, V, j) &= \frac{1}{6}\sum_{i=1}^{2}(-1)^i B_2(x_{3-i}) \left\{\sum_{k=1}^{2} \frac{\det(V_j, V_{j\dotplus k})}{v_{ij\dotplus k}}\right\} \frac{\log v_{ij}}{v_{ij}} \\
\mathfrak{B}_3(x, V, j) &= \frac{-1}{18}\sum_{i=1}^{3}\frac{\log v_{ij}}{v_{ij}}\sum_{k=1}^{2}\frac{1}{v_{ij\dotplus k}}\left\{\frac{\det(\widetilde{V_{i\dotplus 2j\dotplus k}})^2}{\det(\widetilde{V_{i\dotplus 1j\dotplus k}})}B_3(x_{i\dotplus 1})\right. \\
&\quad + \frac{\det(\widetilde{V_{i\dotplus 1j\dotplus k}})^2}{\det(\widetilde{V_{i\dotplus 2j\dotplus k}})}B_3(x_{i\dotplus 2}) + \frac{3\det(\widetilde{V_{i\dotplus 2j\dotplus k}})}{\det(\widetilde{V_{i\dotplus 1j\dotplus k}})}B_2(x_{i\dotplus 1})B_1(x_{i\dotplus 2}) \\
&\quad \left. + \frac{3\det(\widetilde{V_{i\dotplus 1j\dotplus k}})}{\det(\widetilde{V_{i\dotplus 2j\dotplus k}})}B_1(x_{i\dotplus 1})B_2(x_{i\dotplus 2})\right\}.
\end{aligned}$$

Here $\widetilde{V_{ij}}$ denotes the matrix obtained from V by removing the i^{th} row and j^{th} column.

Theorem 1. Assuming that $Re(v_i) > 0$, we have

$$\frac{d}{ds}\sum_{m\in\mathbb{Z}^r_{\geqslant 0}} N\left(\sum_{i=1}^{r}(x_i + m_i)v_{ki}\right)^{-s}\Bigg|_{s=0}$$

$$= \sum_{j=1}^{3}\left\{\log\gamma_r(xV_j, V_j) + \mathfrak{B}_r(x, V, j)\right\}.$$

In the case that the stated positivity assumption is not satified, in other words, the projection of the cone into $\mathbb{C}$ does not belong to right half plane, we employ the folllowing trick. We rotate the given cone to the right half plane by a complex rotation of absolute value 1, and apply the above theorem to the rotated cone, and compensate in the end by choosing a suitable correction factor. That correction factor depends on the choice of the logarithm function. It turns out that the ambiguity of the complex logarithm introduces in the final result a root of unity.

Our final result can be written as

$$\zeta_F'(0, C) = \sum_{j=1}^{3} \Phi_F(C, j),$$

where the index j indicates the three embeddings of the cubic field, and

$$\Phi_F(C,j) = \sum_{k\in S} \Big\{ \log\gamma_{r(k)}(xV_{kj}, V_{kj}) + \mathfrak{B}_{r(k)}(x, V_k, j)$$
$$+ \text{ corretion term due to choice of logarithm } (\in 2\pi i\mathbb{Q})\Big\}.$$

Numerical calculations suggest the following conjecture.

Conjecture 1a: $\Phi_F(C,1) \in \mathbb{Q}\log\eta^{(1)}$.

Conjecture 1b: Let N be the smallest positive rational integer in $\mathfrak{f}$. Then

$$(2N)^3\Phi_F(C,1) \in \mathbb{Z}\log\eta^{(1)}.$$

Remark: Conjecture 1 was verified numerically (40 digits precision) in the case of $F = \mathbb{Q}(\theta)$, $\theta^3 = 2$, and 50 choices of C and $\mathfrak{f}$.

Conjecture 2:

$$\exp\left\{ w_K \left(\Phi_F(C,2) - \Phi_F(C,1)\frac{\log\eta^{(2)}}{\log\eta^{(1)}} \right) \right\} = \zeta\sigma(\varepsilon),$$

where $\sigma(\varepsilon)$ is a Stark unit and ζ is a root of unity depending on the choice of fundamental domain and the choice of the complex logarithm function. Conjecture 2 was verified numerically (40 digits precision) in the case of $F = \mathbb{Q}(\theta)$, $\theta^3 = 2$, and $\mathfrak{f} = (3),\ (4),\ (5)$. In the case $\mathfrak{f} = (3),\ (5)$, we have $\zeta \in K$, but in the case $\mathfrak{f} = (4)$, $Ord(\zeta) = 16$, while $w_K = 4$.

More examples need to be calculated before a prediction can be made regarding the order of ζ.

Based on this initial evidence, I will close with the optimistic prediction that a good formula for the Stark unit ε does exist in all cases of a first order zero of the partial zeta function.