On the $\mathfrak{p}$-adic Stark conjecture at a real place

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1. Introduction and Notations

k – a global field

S – finite set of places of k

- non-empty
- contains all archimedean places

$\zeta(s)$ – S-zeta function of k

(Euler factors for places in S have been removed)

$$\zeta(s) = -\frac{hR}{w} s^n + O(s^{n+1}) \quad \text{near } s = 0\text{, where}$$

$h =$ class number of $\mathcal{O}_S$

$R =$ regulator of $\mathcal{O}_S^*$

$w =$ number of roots of unity in k

$\quad =$ order of torsion subgroup of $\mathcal{O}_S^*$

$n = \#S - 1 =$ rank of $\mathcal{O}_S^*$ (Dirichlet's unit theorem)

Stark's idea: a similar phenomenon should hold for Artin L-functions

K/k – a Galois extension

ψ – the character of a representation of $\mathrm{Gal}(K/k)$

$L(s, \psi)$ – the corresponding Artin L-function

Write $L(s, \psi) = as^m + O(s^{m+1})$ near $s = 0$

"Vague" conjecture:

$a =$ (algebraic number)(transcendental number)

Order of vanishing:

$Y =$ free abelian group on $S(K)$ (places of K lying over places of S)

$X \subseteq Y$ – the submodule of degree 0 elements

$$
\begin{array}{ccccccccc}
 & & & & \sum n_v\, v & \mapsto & \sum n_v & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}
$$

Let χ_X be the character of the $\mathrm{Gal}(K/k)$-module X.

Theorem. *The order of vanishing at $s = 0$ of the Artin L-function $L(s, \psi)$ is $\langle \psi, \chi_X \rangle_{\mathrm{Gal}(K/k)}$.*

"Analytic rank $=$ Algebraic rank."

Abelian case: If K/k is abelian, and χ a 1-dimensional character, then the order of vanishing of $L(s,\chi)$ is the number of places in S that split completely in K_χ (the fixed field of $\ker\chi$).

If $\chi = \chi_0$, the trivial character, then it's 1 less than this (or simply $\#S - 1$).

2. Rank 1 abelian Stark conjecture

K/k – an abelian extension of global fields

S – finite set of places of k, as before, but also

- S contains all places ramified in K
- S contains a place, $\mathfrak{p}$, that splits completely in K. Let

$\mathfrak{P}$ be a place of K lying over $\mathfrak{p}$.

These conditions on S imply that $L(s,\chi)$ vanishes to order at least 1 at $s = 0$. So we are interested in the coefficient of s^1 in the Taylor series.

Conjecture. *There is* $\varepsilon \in K$ *(the "Stark unit") such that*

(a) $L'(0,\chi) = -\frac{1}{w_K} \sum_{g \in \mathrm{Gal}(K/k)} \chi(g) \log |\varepsilon^g|_{\mathfrak{P}}$

for all characters χ *on* $\mathrm{Gal}(K/k)$

(b) $K(\varepsilon^{1/w_K})$ *is abelian over* k.

(c) ε *is an* $S(K)$*-unit (and more)*

Condition (c) actually specifies $|\varepsilon|_{\mathfrak{Q}}$ at all places. Thus ε is determined up to a root of unity, but no further.

Classical case:

$k = \mathbb{Q}$, $K = \mathbb{Q}(\zeta_m)$

p is a prime $\equiv 1 \bmod m$

$S = \{\infty, p\} \cup \mathrm{Supp}(m)$

$\lambda = \varepsilon^{1/m} \sim$ Gauss sum $\in \mathbb{Q}(\zeta_{mp})$

3. Refined $\mathfrak{p}$-adic conjecture

Idea: $\mathfrak{p}$-adic understanding of the Stark unit

Let K/k, S, $\mathfrak{p}$, $\mathfrak{P}$, ε be as before.

Let L be an extension of K, abelian over k, and unramified outside of S, and let $G = \mathrm{Gal}(L/k)$ and $H = \mathrm{Gal}(L/K)$.

$\theta =$ "Stickelberger" element in $\mathbb{C}[G]$ determined by

$$\chi(\theta) = L(0, \bar{\chi}) \quad \text{for all characters } \chi \in \widehat{G}$$

The following are known:

- θ has rational coefficients. (Siegel, Weil)

- The denominators of its coefficients are bounded. In fact, if $A \in \mathbb{Z}[G]$ annihilates the G-module $\mu(L)$, then $A\theta \in \mathbb{Z}[G]$. (Barsky, Cassou-Noguès, Deligne-Ribet, Weil)

We'll consider $A = \varphi_{\mathfrak{q}} - \mathbf{N}\mathfrak{q}$, where $\mathfrak{q} \notin S$ does not divide w_L.

The "modified Stark unit"

Let $\lambda = \varepsilon^{1/w_K}$. The extension $L(\lambda)$ is abelian over k. Define $\varepsilon_{\mathfrak{q}}$ to be $\lambda^{\widetilde{\varphi_{\mathfrak{q}}} - \mathbf{N}\mathfrak{q}}$. It's easy to show that

- $\varepsilon_{\mathfrak{q}} \in K$
- $\varepsilon_{\mathfrak{q}}$ is independent of the choice of ε.

Conjecture. *Write $(\varphi_{\mathfrak{q}} - \mathbf{N}\mathfrak{q})\theta = \sum_G n_g \cdot g$. Then $r_{\mathfrak{P}}(\varepsilon_{\mathfrak{q}}) = \prod_H h^{n_h}$, where $r_{\mathfrak{P}}$ is the local reciprocity map at $\mathfrak{P}$ for the extension L/K.*

Classical case: ($k = \mathbb{Q}$) $\mathfrak{p}$-adic conjecture is known (Gross-Koblitz)

Also proved in the function field case by Hayes.

4. My contributions

L/k a CM-extension, i.e. k is totally real, L totally complex, and $K =$ maximal totally real subfield satisfies $[L : K] = 2$. (We also assume that $k \neq \mathbb{Q}$.) $\mathfrak{p}$ is a real place of k.

Stark's (rank 1 abelian) conjecture holds trivially. In most cases, all the L-functions vanish to higher order, so we may take $\varepsilon = 1$.

The $\mathfrak{p}$-adic conjecture is non-trivial! It comes down to the evenness of the coefficients of $(\varphi_{\mathfrak{q}} - \mathbf{N}\mathfrak{q})\theta$. This is proved using the 2-adic congruences of Deligne-Ribet.

In the special case that k is a real quadratic field and L is the narrow Hilbert class field, the Stark unit is not trivial, it is a power of the fundamental unit of k. This case matches closely the "exceptional case" isolated by Deligne-Ribet, when they obtained a weaker 2-divisibility. In this situation, the coefficients of $(\varphi_{\mathfrak{q}} - \mathbf{N}\mathfrak{q})\theta$ are all even or are all odd, depending upon $\mathfrak{q}$, and the $\mathfrak{p}$-adic conjecture holds in this situation.

Let $G_0 \subseteq G$ be the subgroup generated by complex conjugations. It's an elementary abelian 2-group of rank $\leq [k : \mathbb{Q}]$. We consider the case when the rank is strictly less than $[k : \mathbb{Q}]$.

Theorem. *Suppose that*

(a) *The rank of G_0 is $< [k : \mathbb{Q}]$,*

(b) $\mathfrak{p}$ *is a real place of k which ramifies in L, and*

(c) $\#S \geq 3$.

Then the $\mathfrak{p}$-adic conjecture for $L/K/k$ implies that the Stark unit (chosen to be positive at $\mathfrak{P}$) is a square in K.

Consequences:

The abelian condition in Stark's conjecture holds automatically, because $w_K = 2$.

Perhaps the denominator (w_K) in the conjecture can be made smaller in some situations like this.

Situation of Dummit and Hayes

$k =$ totally real number field of odd degree over $\mathbb{Q}$

$L =$ narrow Hilbert class field, $K =$ fixed field of complex conjugation at an archimedean place.

In their case, all L functions from L/k vanish, so the Stickelberger element does also.

In our case, we use the Deligne-Ribet 2-adic congruences to show that all coefficients of $(\varphi_{\mathfrak{p}} - \mathbf{N}\mathfrak{p})\theta$ are even.

5. A computational example

$k = \mathbb{Q}(\alpha)$, where $\alpha^4 - 6\alpha^2 - 3\alpha + 3 = 0$. $\quad (d_k = 3^3 \cdot 367)$

∞_1 is the archimedean place $\alpha \mapsto -1.20752035\ldots$

$\mathfrak{p}$ is the prime ideal $(6\alpha^2 - 5\alpha + 1)$ which divides 109.

$L =$ ray class field modulo $\mathfrak{p}\infty_1\infty_2\infty_3\infty_4$

$K =$ ray class field modulo $\mathfrak{p}\infty_2\infty_3\infty_4$

$\mathrm{Gal}(L/k) \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Gal}(K/k) \cong \mathbb{Z}/4\mathbb{Z}$

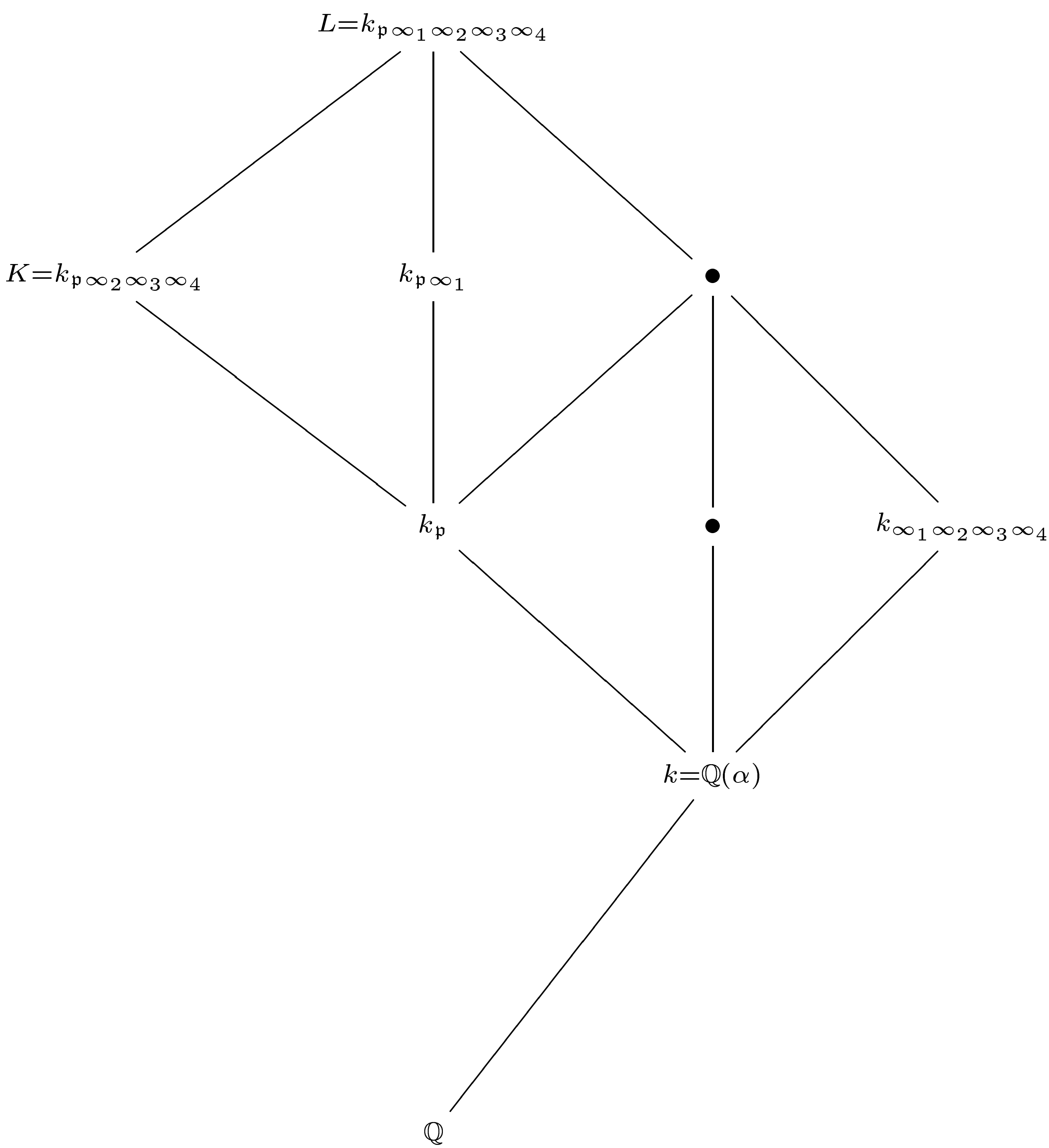
Pari/GP calculates the values of $L'(0,\chi)$ for characters on $\mathrm{Gal}(K/k)$; they are

$$L'(0,\chi) = -5.2799158505\ldots - i9.4760111099\ldots$$

$$L'(0,\chi^3) = -5.2799158505\ldots + i9.4760111099\ldots$$

Using this, we identify the minimal polynomial, $f(X)$, of ε over k. Over k, the polynomial $f(X)$ is irreducible, but $f(X^2)$ factors as $g(X)g(-X)$.

Lattice of fields



$$k_{\infty_1\infty_2\infty_3\infty_4} = k(\sqrt{-(\alpha+2)}) = k(\sqrt{-3}).$$

$$k_{\mathfrak{p}} = k(\sqrt{6\alpha^2 - 5\alpha + 1})$$

$k_{\mathfrak{p}\infty_2\infty_3\infty_4} = k(\sqrt{c_0 + c_1\sqrt{6\alpha^2 - 5\alpha + 1}})$, where

$c_0 = 5954\alpha^3 - 7190\alpha^2 - 27042\alpha + 14784$ and

$c_1 = 1446\alpha^3 - 1769\alpha^2 - 6558\alpha + 3660.$

The Stark unit is

$$\left(\frac{b_0 + b_1\sqrt{6\alpha^2 - 5\alpha + 1} + 2\sqrt{c_0 + c_1\sqrt{6\alpha^2 - 5\alpha + 1}}}{4}\right)^2$$

where c_0 and c_1 are as above, and

$b_0 = 58\alpha^3 - 70\alpha^2 - 264\alpha + 142$, and

$b_1 = 18\alpha^3 - 10\alpha^2 - 90\alpha - 12.$