$K \subset L$: finite abelian extension of global fields
$G = \mathrm{Gal}(L/K)$

S: finite non-empty set of places of K, containing all archimedean ones and ones ramified in L

T: finite non-empty set of places of K, disjoint from S, such that $U_{S,T} = \{x \in U_S \mid x \equiv 1 \pmod{v}, \forall v \in T\}$ is *torsion-free*

For each complex character $\chi \in \widehat{G}$, the associated modified L-function $L_{S,T}(\chi, s)$ is defined as an infinite product

$$\prod_{v \in T} (1 - \chi(\sigma_v) \boldsymbol{N} v^{1-s}) \prod_{v \notin S} (1 - \chi(\sigma_v) \boldsymbol{N} v^{-s})^{-1}$$

There is a unique $\mathbb{C}[G]$-valued function $\theta(s)$ which satisfies

$$\chi(\theta(s)) = L_{S,T}(\chi, s) \quad \forall \chi \in \widehat{G}, \forall s \in \mathbb{C} \setminus \{1\}$$

Theorem. $\theta(1-n) \in \mathbb{Z}[G]$ *for all integers* $n \geq 1$.

The augmentation ideal I_G is defined by the short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_G & \longrightarrow & \mathbb{Z}[G] & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & & & g & \longmapsto & 1 & & \end{array}$$

Let $n = |S| - 1 =$ the order of zero of $\zeta_{K,S,T}(s)$ at $s = 0$, and let

$$h_{S,T} = \#\frac{\{\text{fractional ideals of } O_S, \text{coprime to } T\}}{\{(x) | x \equiv 1 \pmod{v}, \forall v \in T\}}$$

Let Y be the free $\mathbb{Z}$-module generated by the places $v \in S$ and define X by

$$\begin{aligned} 0 \longrightarrow X \longrightarrow Y \longrightarrow \mathbb{Z} \longrightarrow 0 \\ v \longmapsto 1 \end{aligned}$$

We define a homomorphism

$$\begin{aligned} \lambda_{S,T} : U_{S,T} &\longrightarrow (I_G/I_G^2) \otimes X \\ \varepsilon &\longmapsto \sum_{v \in S} (f_v(\varepsilon) - 1) \cdot v. \end{aligned}$$

where f_v is the local Artin reciprocity map at v.

Conjecture. *(Gross)* $\theta_{S,T} := \theta(0) \in I_G^n$, *and that*

$$\theta_{S,T} \equiv h_{S,T} \cdot \det \lambda_{S,T} \pmod{I_G^{n+1}}$$

Let $G = G_1 \times G_2 \times \cdots \times G_m$,
$H_i = G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_m$, and let $\phi_i : \mathbb{Z}[G] \longrightarrow \mathbb{Z}[H_i]$ be the map induced by natural projection.

Theorem. *Suppose $\alpha \in \mathbb{Z}[G]$ such that $\phi_i(\alpha) \in I^n_{H_i}$ for $i = 1, \ldots, n$. Then $\alpha \in I^n_G$. (We assume $m \geq n$.)*

Corollary. *Choose a prime number l. Suppose K is a global function field of characteristic $p \neq l$ such that*

- *its class number is prime to l.*
- *it does not contain a primitive l-th root of 1.*

Then Gross's conjecture holds for all L/K whose Galois group is an abelian l-group.