Refinement of Gross' Conjecture to non-maximal orders

K :=a quad. im. field, class number 1

$\mathcal{O}_K$:=ring of integers of K

Fix $K \hookrightarrow \mathbb{C}$.

$\mathcal{O}$:=an order in K

E := an elliptic curve def. over abelian F/K,
with End$(E) = \mathcal{O}$

G :=Gal(F/K)

S_∞ :=infinite places of F, equivalently
embeddings $\iota : F \to \mathbb{C}$
agreeing with $K \hookrightarrow \mathbb{C}$

ψ := the Grössencharacter of E/F

$$H_{\mathsf{B}}(E) := \bigoplus_{v \in S_\infty} H^1(E(F_v), 2\pi i \cdot \mathbb{Q})$$

Use $H^1(E, \mathcal{O}_E) \simeq H^0(E, \Omega_{E/F})^\vee =$ coLie$(E/F)^\vee$
where "$\vee$" is the dual.

For any $\mathbb{Q}$-vector space W we write $W_{\mathbb{R}} := W \otimes_{\mathbb{Q}} \mathbb{R}$.

We use the comparison isomorphism

$$I_E : H_{\mathsf{B}}(E)_{\mathbb{R}} \xrightarrow{\simeq} H^1(E, \mathcal{O}_E)_{\mathbb{R}},$$

and its determinant

$$\begin{aligned} c^+(E) &:= \det I_E \\ &\in \det_{K_{\mathbb{R}}}(H_{\mathsf{B}}(E)_{\mathbb{R}})^{-1} \otimes_{K_{\mathbb{R}}} \det_{K_{\mathbb{R}}}(H^1(E, \mathcal{O}_E)_{\mathbb{R}}). \end{aligned}$$

Fix a prime number $p \neq 2, 3$. Write

$$K_p := K \otimes_{\mathbb{Q}} \mathbb{Q}_p$$

and

$$\mathcal{O}_p := \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

$\mathcal{O}_{F,S} := \mathcal{O}_F$ with primes $\in S - S_\infty$ inverted.

$$\begin{tikzcd}
R\Gamma_{\mathsf{c}}(\mathcal{O}_{F,S}, V_p(E)) \arrow[d] \\
R\Gamma(\mathcal{O}_{F,S}, V_p(E)) \arrow[d] \\
\prod_{v\in S} R\Gamma(F_v, V_p(E))
\end{tikzcd}$$

Assume $L(\overline{\psi}, 1) \neq 0$, so that $\mathrm{rank}_{\mathbb{Z}} E(F) = 0$. Then one can construct a minimal isomorphism

$$\begin{aligned}\vartheta_p : &\det_{K_p} R\Gamma_c(\mathcal{O}_{F,S}, V_p(E)) \\ &\simeq \det_{K_p}(\mathbb{Q}_p \otimes_{\mathbb{Q}} H^1(E, \mathcal{O}_E)) \otimes \det_{K_p}(\mathbb{Q}_p \otimes_{\mathbb{Q}} H_{\mathsf{B}}(E))^{-1}.\end{aligned}$$

Conjecture 0 (Burns – Flach) *There is an isomorphism of invertible $\mathcal{O}_p$-modules*

$$\vartheta_p \left(\det_{\mathcal{O}_p} R\Gamma_c(\mathcal{O}_{F,S}, T_p(E)) \right) \simeq \mathcal{O}_p \cdot L(\overline{\psi}, 1)^{-1} c^{+}(E).$$

The Weil Restriction

Use notation analogous to that for E.

Note ∃ isogeny $E \to$ elliptic curve E_0 over F with $\mathrm{End}_F E_0 \simeq \mathcal{O}_K$. Assume E_0 defined over K and let φ be the Grössencharacter of E_0/K. Then $\psi = \mathbf{N} \circ \varphi$. Put

$$\begin{aligned} B :=& \mathrm{Res}_K^F E, \\ \mathcal{R} :=& \mathrm{End}_K B, \\ R :=& \mathcal{R} \otimes \mathbb{Q} \simeq K[G]. \end{aligned}$$

Define the $R \otimes_{\mathbb{Q}} \mathbb{R}$-valued L-function of B

$$\begin{aligned} L(B,s) :=& \, (L(\overline{\varphi\chi}, s))_{\chi \in \widehat{G}} \\ & \in \prod_{\chi \in \widehat{G}} \mathbb{C} \simeq \mathbb{C}[G] \simeq K[G] \otimes_{\mathbb{Q}} \mathbb{R} \simeq R \otimes_{\mathbb{Q}} \mathbb{R}. \end{aligned}$$

We have, for B, a conjecture analogous to Conjecture 0.

Conjecture 1 *There is an isomorphism of invertible $\mathcal{R}_p$-modules*

$$\vartheta_p \left(\det_{\mathcal{R}_p} R\Gamma_c(\mathcal{O}_{K,S}, T_p(B)) \right) \simeq \mathcal{R}_p \cdot L(B,1)^{-1} c^{+}(B).$$

Taking norms from $\mathcal{R}_p$ to $\mathcal{O}_p$ returns Conjecture 0.

We reformulate the Conjecture once more. Replacing the maps and exact sequences implicit in the construction of ϑ_p by their duals, we find that Conjecture 1 is equivalent to

Conjecture 2 *The image of*

$$\Delta_{\mathcal{R}_p}(\mathcal{O}_{K,S}, T_p(B))$$

under

$$\vartheta_p^\vee : \Delta_{\mathcal{R}_p}(\mathcal{O}_{K,S}, T_p(B)) \simeq H^0(B, \Omega^1) \otimes_{\mathcal{R}_p} H_{\mathrm{B}}(B)^{-1}$$

is generated by

$$L(B,1)^{-1} c^+(B)^\vee.$$

Here

$$\begin{aligned} &\Delta_{\mathcal{R}_p}(\mathcal{O}_{K,S}, T_p(B)) \\ &:= \left(\det_{\mathcal{R}_p} R\Gamma(\mathcal{O}_{K,S}, T_p(B))\right)^{-1} \\ &\quad \otimes_{\mathcal{R}_p} \left(\det_{\mathcal{R}_p} R\Gamma(K_{\mathbb{R}}, T_p(B)(-1))\right)^{-1} \end{aligned}$$

is dual to $\det_{\mathcal{R}_p} R\Gamma_c(\mathcal{O}_{K,S}, T_p(B))$, by (a derived-category version of) Tate-Poitou duality.

We now pass to a situation universal with respect to invertible sheaves over p-adic rings such as $T_p(B)$ over $\mathcal{R}_p$.

$$K^{\mathfrak{m}} = \text{ray class field modulo } \mathfrak{m}$$

$$\mathfrak{f} = \text{conductor of } B$$

Choose $\mathfrak{g} \subseteq \mathfrak{f}$ prime to p so

$$F \subseteq K^{\mathfrak{g}\mathfrak{p}^n} \text{ for } n \gg 0$$

$$\Lambda_\infty := \mathbb{Z}_p[[\mathrm{Gal}(K^{\mathfrak{g}p^\infty}/K)]]$$

$$\mathcal{F}_\infty := \varprojlim_n f_{n,*} f_n^* \mathbb{Z}_p(1)_{\mathrm{Spec}\mathcal{O}_{F,S}}$$

$$(f_n = \text{canonical } \mathrm{Spec}\mathcal{O}_{K^{\mathfrak{g}p^n},S} \to \mathrm{Spec}\mathcal{O}_{F,S}).$$

Action of G on $T_p(B)(-1)$ is given by character $G \twoheadrightarrow \mathrm{Gal}(K^{\mathfrak{g}p^\infty}/K) \longrightarrow \mathcal{R}^\times$, inducing $\Lambda_\infty \to \mathcal{R}_p$, so that

$$T_p(B) \simeq \mathcal{F}_\infty \otimes_{\Lambda_\infty} \mathcal{R}_p$$

and

$$\Delta_{\Lambda_\infty}(\mathcal{O}_{K,S}, \mathcal{F}_\infty) \otimes_{\Lambda_\infty} \mathcal{R}_p \simeq \Delta_{\mathcal{R}_p}(\mathcal{O}_{K,S}, \mathcal{T}_p(B)).$$

There are two parts to the rest of the proof.

•First, we construct an $\mathcal{R}$-basis $z_{\mathcal{R}}(\mathcal{O}_{K,S}, T_p(B))$ of $\Delta_{\mathcal{R}}(\mathcal{O}_{K,S}, T_p(B))$.
This is done by constructing a Λ_∞-basis $z_{\Lambda_\infty}(\mathcal{O}_{K,S}, \mathcal{F}_\infty)$ of $\Delta_{\Lambda_\infty}(\mathcal{O}_{K,S}, \mathcal{F}_\infty)$, using elliptic units. The proof uses Rubin's Main Conjecture.

• Second, we prove

$\vartheta_p^\vee(z_{\mathcal{R}}(\mathcal{O}_{K,S}, T_p(B))) \in (H^0(B, \Omega^1) \otimes_A H_{\mathrm{B}}^{-1}) \otimes \mathbb{Q}_p$ is equal to $L(B, 1)^{-1} c^+(B)^\vee$. This uses the explicit reciprocity law.

Theorem 3 *Main Conjecture.*

$$\mathrm{char}(A_\infty) = \mathrm{char}(\overline{\mathcal{E}}_\infty/\overline{\mathcal{C}}_\infty),$$

where

•A_∞ *is the* p*-part of the ideal class group of* $K_{\mathfrak{g}p^\infty}$*,*

•$\overline{\mathcal{E}}_\infty$ *are the global units, and*

•$\overline{\mathcal{C}}_\infty$ *are the elliptic units.*

(The last two groups are defined, precisely, as the inverse limits, with respect to the norm maps, of the corresponding groups for finite subextensions of $K_{\mathfrak{g}p^\infty}/K$*.)*

The above modules are related to our situation by

- the canonical isomophism

$$\overline{\mathcal{E}}_\infty(p) \simeq \varprojlim_n \mathcal{O}_{K^{\mathfrak{g}p^n}}\left[\frac{1}{p}\right]^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq H^1(\mathcal{O}_K\left[\frac{1}{p}\right], \mathcal{F}_\infty),$$

- the exact sequence

$$\begin{tikzcd}
A_\infty(p) \arrow[d, equal] \\
\varprojlim_n \left(\mathrm{Pic}\left(\mathcal{O}_{K^{\mathfrak{g}p^n}}\left[\frac{1}{p}\right]\right)\right) \otimes_{\mathbb{Z}} \mathbb{Z}_p \arrow[d, "\phi"'] \\
H^2(\mathcal{O}_K\left[\frac{1}{p}\right], \mathcal{F}_{K^{\mathfrak{g}p^\infty}}) \arrow[d] \\
\mathrm{Tr}^0\left(\bigoplus_{v|p} \mathbb{Z}\right) \otimes_{\mathbb{Z}} \mathbb{Z}_p \arrow[d] \\
0
\end{tikzcd}$$

(where v denotes a place of $K^{\mathfrak{g}p^\infty}$), and

- the equality

$$\overline{\mathcal{C}}_\infty = z_{\Lambda_\infty}(\mathcal{O}_{K,S}, \mathcal{F}_\infty) \cdot \Lambda_\infty.$$