The Equivariant Tamagawa Number Conjecture: Background and Formalism

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- The Tamagawa Number Conjecture in the formulation of Fontaine and Perrin-Riou

- Its Equivariant Refinement

- Determinant Functors: Some Algebra

Joint with David Burns

$$X \to \mathrm{Spec}(\mathbb{Q}) \text{ smooth projective}$$

$$M = h^i(X)(j) \qquad \text{"}motive\text{"} \qquad i, j \in \mathbb{Z}$$

$$M_l = H^i_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_l)(j) \quad G_{\mathbb{Q}}\text{-representation}$$

$$P_p(T) = \det(1 - \mathrm{Fr}_p^{-1} \cdot T | M_l^{I_p}) \in \mathbb{Q}[T]$$

$$L(M, s) = \prod_p P_p(p^{-s})^{-1}$$

$$L(M, s) = L^*(M) s^{r(M)} + \cdots \quad \text{expansion at } s = 0$$

Aim: Describe $L^*(M) \in \mathbb{R}^\times$ and $r(M) \in \mathbb{Z}$ (as in the Conjecture of BSD, or the analytic class number formula)

Periods and Regulators

$$M_B = H^i(X(\mathbb{C}), \mathbb{Q})(j), \quad M_{dR} = H^i_{dR}(X/\mathbb{Q})(j)$$

$$H^0_f(M), H^1_f(M) \quad \text{Motivic cohomology}$$

The period isomorphism $M_B \otimes_{\mathbb{Q}} \mathbb{C} \cong M_{dR} \otimes_{\mathbb{Q}} \mathbb{C}$ induces a map

$$\alpha_M : M_B^+ \otimes_{\mathbb{Q}} \mathbb{R} \to (M_{dR}/ \operatorname{Fil}^0 M_{dR}) \otimes_{\mathbb{Q}} \mathbb{R}$$

Conjecture Mot$_\infty$: There exists an exact sequence

$$0 \to H^0_f(M)_{\mathbb{R}} \xrightarrow{c} \ker(\alpha_M) \to H^1_f(M^*(1))^*_{\mathbb{R}} \xrightarrow{h}$$
$$H^1_f(M)_{\mathbb{R}} \xrightarrow{r} \operatorname{coker}(\alpha_M) \to H^0_f(M^*(1))^*_{\mathbb{R}} \to 0$$

c=cycle class map, h=height pairing, r=Beilinson regulator.

Conjecture 1 (Vanishing Order):

$$r(M) = \dim_{\mathbb{Q}} H^1_f(M^*(1)) - \dim_{\mathbb{Q}} H^0_f(M^*(1))$$

Define a $\mathbb{Q}$-vector space of dimension 1

$$\begin{aligned}\Xi(M) := {} & \det_{\mathbb{Q}}(H^0_f(M)) \otimes \det^{-1}_{\mathbb{Q}}(H^1_f(M)) \\ & \otimes \det_{\mathbb{Q}}(H^1_f(M^*(1))^*) \otimes \det^{-1}_{\mathbb{Q}}(H^0_f(M^*(1))^*) \\ & \otimes \det^{-1}_{\mathbb{Q}}(M_B^+) \otimes \det_{\mathbb{Q}}(M_{dr}/\operatorname{Fil}^0)\end{aligned}$$

Conjecture Mot_∞ induces an isomorphism

$$\vartheta_\infty : \Xi(M) \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}$$

Conjecture 2 (Rationality):

$$\vartheta_\infty^{-1}(L^*(M)) \in \Xi(M) \otimes 1$$

Galois cohomology

Define for each prime p a complex $R\Gamma_f(\mathbb{Q}_p, M_l)$

$$= \begin{cases} M_l^{I_p} \xrightarrow{1-\mathrm{Fr}_p} M_l^{I_p} & l \neq p \\ D_{cris}(M_l) \xrightarrow{(1-\mathrm{Fr}_p,\pi)} \begin{array}{l} D_{cris}(M_l) \oplus \\ D_{dR}(M_l)/\,\mathrm{Fil}^0 \end{array} & l = p \end{cases}$$

Exact triangle

$$R\Gamma_f(\mathbb{Q}_p, M_l) \to R\Gamma(\mathbb{Q}_p, M_l) \to R\Gamma_{/f}(\mathbb{Q}_p, M_l)$$

Let S be a finite set of primes containing l, ∞ and primes of bad reduction.

$$R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_l) \to R\Gamma(\mathbb{Z}[\frac{1}{S}], M_l) \to \bigoplus_{p \in S} R\Gamma(\mathbb{Q}_p, M_l)$$

$$R\Gamma_f(\mathbb{Q}, M_l) \to R\Gamma(\mathbb{Z}[\frac{1}{S}], M_l) \to \bigoplus_{p \in S} R\Gamma_{/f}(\mathbb{Q}_p, M_l)$$

$$R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_l) \to R\Gamma_f(\mathbb{Q}, M_l) \to \bigoplus_{p \in S} R\Gamma_f(\mathbb{Q}_p, M_l)$$

Conjecture Mot$_l$: There are natural isomorphisms $H^0_f(M)_{\mathbb{Q}_l} \cong H^0_f(\mathbb{Q}, M_l)$ (cycle class map) and $H^1_f(M)_{\mathbb{Q}_l} \cong H^1_f(\mathbb{Q}, M_l)$ (Chern class map).

Remark: $H^i_f(\mathbb{Q}, M_l) \cong H^{3-i}_f(\mathbb{Q}, M_l^*(1))^*$.

The last exact triangle of the previous page induces an isomorphism

$$\vartheta_l : \det_{\mathbb{Q}_l} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_l) \cong \Xi(M) \otimes_{\mathbb{Q}} \mathbb{Q}_l$$

Let $T_l \subset M_l$ be any $G_{\mathbb{Q}}$-stable $\mathbb{Z}_l$-lattice.

Conjecture 3 (Integrality):

$$\vartheta_l(\det_{\mathbb{Z}_l} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T_l)) = \mathbb{Z}_l \cdot \vartheta_\infty^{-1}(L^*(M)^{-1})$$

This conjecture (for all l) determines $L^*(M)$ up to sign.

Equivariant Generalization

In many situations one has 'extra symmetries', more precisely there is a semisimple, finite dimensional $\mathbb{Q}$-algebra A acting on M.

Examples:

- X an abelian variety, $A = \mathrm{End}(X) \otimes \mathbb{Q}$
- $X = X' \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(K)$, $K/\mathbb{Q}$ Galois with group G, $A = \mathbb{Q}[G]$

If A is **commutative** (i.e. a product of number fields) one can construct $L({}_AM, s)$, ϑ_∞, ϑ_l as before using determinants over A, $A \otimes \mathbb{R}$, $A \otimes \mathbb{Q}_l$.

One gets refinements of Conjectures 1 and 2 with $r({}_AM) \in H^0(\mathrm{Spec}(A \otimes \mathbb{R}), \mathbb{Z})$ and $L^*({}_AM) \in (A \otimes \mathbb{R})^\times$.

Taking "Norms form A to $\mathbb{Q}$" gives the original conjectures.

Somewhat more interesting is the generalization of Conjecture 3. There are many $\mathbb{Z}$-orders $\mathfrak{A} \subseteq A$ unlike in the case $A = \mathbb{Q}$.

Assume that there is a **projective** $G_{\mathbb{Q}}$-stable $\mathfrak{A}_l := \mathfrak{A} \otimes \mathbb{Z}_l$ lattice $T_l \subset M_l$.

Then $R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T_l)$ is a perfect complex of $\mathfrak{A}_l$-modules and $\det_{\mathfrak{A}_l} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T_l)$ is an invertible (i.e. free rank 1) $\mathfrak{A}_l$-module. The generalized Conjecture 3 says that $\vartheta_l^{-1}\vartheta_\infty^{-1}(L^*({}_AM)^{-1})$ is a basis.

Some proven cases with non-maximal $\mathfrak{A}$:

X a CM elliptic curve, $A = \mathrm{End}(X)$ (Colwell)

$X = \mathrm{Spec}(K)$, $K/\mathbb{Q}$ abelian, $\mathfrak{A} = \mathbb{Z}[G]$ (Burns-Greither)

$X = X_0(N)$, N prime, $M = \mathrm{End} h^1(X)$, $\mathfrak{A} =$ Hecke Algebra (Qiang Lin)

Determinant Functors

R commutative ring

P finitely generated projective R-module

$\mathrm{det}_R(P) := \bigwedge^{\mathrm{rank}_R(P)} P$ invertible R-module

$\mathrm{rank}_R(P) \in H^0(\mathrm{Spec}(R), \mathbb{Z})$ locally constant

The pair $\mathrm{Det}_R(P) := (\mathrm{det}_R(P), \mathrm{rank}_R(P))$ is called a **graded** invertible R-module. Let $\mathrm{Inv}(R)$ be the category of such with isomorphisms.

Functor $\mathrm{Det}_R : (\mathrm{PrMod}(R), is) \to \mathrm{Inv}(R)$.

A short exact sequence

$$0 \to P_1 \to P_2 \to P_3 \to 0$$

induces a (functorial) isomorphism

$$\mathrm{Det}_R(P_2) \cong \mathrm{Det}_R(P_1) \otimes \mathrm{Det}_R(P_3).$$

Det_R can be extended to perfect complexes and isomorphisms of such.

Inv(R) is an example of a **Picard category**:

• All morphisms are isomorphism.

• There is a bifunctor $(L, M) \mapsto L \boxtimes M$ with unit object $\mathbf{1}$, inverses, associativity and commutativity constraint.

Definition: Let R be any ring. A **determinant functor** is a Picard category $\mathcal{P}$, a functor D : (PrMod(R), is) $\to \mathcal{P}$, and functorial isomorphisms $D(P_2) \cong D(P_1) \boxtimes D(P_3)$ for short exact sequences + some conditions.

Theorem (Deligne) For any ring R there is a universal determinant functor

$$D_R : (\text{PrMod}(R), is) \to V(R).$$

$V(R)$ is called the category of virtual objects of R.

This is a categorical version of K_0.

Indeed D_R induces isomorphisms

$$K_0(R) \xrightarrow{\sim} \pi_0(V(R)) := \begin{cases} \text{Iso.classes of objects} \\ \text{with product induces by } \boxtimes \end{cases}$$

$$K_1(R) \xrightarrow{\sim} \pi_1(V(R)) := \mathrm{Aut}_{V(R)}(\mathbf{1})$$

If R is commutative we have a (tensor) functor $V(R) \to \mathrm{Inv}(R)$ by universality. This functor induces surjections

$$K_0(R) \twoheadrightarrow \mathrm{Pic}(R) \oplus H^0(\mathrm{Spec}(R), \mathbb{Z}) = \pi_0(\mathrm{Inv}(R))$$

$$K_1(R) \twoheadrightarrow R^\times = \pi_1(\mathrm{Inv}(R))$$

For $R = A, \mathfrak{A}, A_l, \mathfrak{A}_l, A \otimes \mathbb{R}$ these are **isomorphisms** (except for $K_1(\mathfrak{A}) \to \mathfrak{A}^\times$).

Now we can generalize to arbitrary (semisimple) algebras A.

$\Xi(M)$ is an object of $V(A)$.

$\vartheta_\infty : \Xi(M) \otimes_A A_{\mathbb{R}} \cong \mathbf{1}_{V(A_{\mathbb{R}})}$

$\Psi_l := D_{\mathfrak{A}_l}(R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T_l))$ is an object of $V(\mathfrak{A}_l)$.

$\vartheta_l : \Psi_l \otimes_{\mathfrak{A}_l} A_l \cong \Xi(M) \otimes_A A_l$

Under some weak assumption on integral lattices in motivic cohomology $(\Xi(M), \Psi_l, \vartheta_l)$ gives an object of

$$\mathbb{V}(\mathfrak{A}) := V(\mathfrak{A} \otimes \hat{\mathbb{Z}}) \times_{V(A \otimes \hat{\mathbb{Z}})} V(A)$$

hence a class in $\pi_0(\mathbb{V}(\mathfrak{A})) \cong K_0(\mathfrak{A})$.

Similarly $(\Xi(M), \Psi_l, \vartheta_l, \vartheta_\infty)$ gives a class

$$R(\mathfrak{A}, M) \in K_0(\mathfrak{A}, \mathbb{R})$$

where $K_0(\mathfrak{A}, \mathbb{R})$ is the relative K_0 for $\mathfrak{A} \to A_{\mathbb{R}}$.

Conjecture 3 (general form):

$$\widehat{\delta}(L^*({}_A M)) + R(\mathfrak{A}, M) = 0$$

$$\begin{array}{ccccccc}
K_1(A_{\mathbb{R}}) & \xrightarrow{\delta} & K_0(\mathfrak{A}, \mathbb{R}) & \longrightarrow & K_0(\mathfrak{A}) & \longrightarrow & K_0(A_{\mathbb{R}}) \\
{\scriptstyle red.Norm}\downarrow & \nearrow_{\exists \widehat{\delta}} & & & & & \\
\zeta(A_{\mathbb{R}})^{\times} \ni L^*({}_A M) & & & & & &
\end{array}$$

Theorem (B-F) Let L/K be Galois with group G, $M = H^0(\mathrm{Spec}(L))$, $\mathfrak{A} = \mathbb{Z}[G]$. Then the image of Conj. 3 in $K_0(\mathbb{Z}[G])$ coincides with Chinburg's conjecture

$$\omega(L/K) + \Omega(L/K, 3) = 0.$$

For $\mathfrak{A}$ a maximal order we recover Chinburg's strong Stark conjecture.

Genuine non-abelian examples? $\mathfrak{A} = \mathbb{Z}[A_5]$? Want nontrivial kernel of $K_0(\mathbb{Z}[A_5], \mathbb{R}) \to K_0(\mathfrak{M}, \mathbb{R})$.