The quantum dilogarithm and quantization of cluster varieties

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Ever since quantum mechanics was discovered, the question what exactly quantization should mean mathematically attracted a lot of attention. Most optimistically, given a symplectic manifold X with an automorphism group Γ, one should have:

1. A projective representation of Γ in a Hilbert space H, of functional dimension $\frac{1}{2}\dim(X)$.
2. A non-commutative deformation A of algebra of functions on X, equipped with a Γ-action.
3. A Γ-equivariant $*$-representation of the $*$-algebra A in H.

Classical examples are given by unitary representations assigned to coadjoint orbits of Lie groups, including a more sophisticated Weils representation of the symplectic group. Cluster varieties are Poisson varieties equipped with an action of an automorphisms group, the cluster modular group. Examples are provided by the moduli spaces of representations of the fundamental group of a punctured surface S to a simple Lie group. The cluster modular group there is the Teichmuller group of S.

We show that cluster varieties admit a quantization in the above sense, where Γ is the cluster modular group. The key role in the construction is played by the quantum dilogarithm. In particular we get a series of infinite dimensional unitary projective representations of the Teichmuller group. We conjecture that they give rise to an unfinite dimensional modular functor. This is a joint work with V. Fock.