

# THE SPACE OF EMBEDDED MINIMAL SURFACES OF FIXED GENUS IN A 3-MANIFOLD III; PLANAR DOMAINS

TOBIAS H. COLDING AND WILLIAM P. MINICOZZI II

## 0. INTRODUCTION

This paper is the third in a series where we describe the space of all embedded minimal surfaces of fixed genus in a fixed (but arbitrary) closed 3-manifold. In [CM3]–[CM5] we describe the case where the surfaces are topologically disks on any fixed small scale (in fact, Corollary III.3.5 below is used in [CM5]). To describe general planar domains (in [CM6]) we need in addition to the results of [CM3]–[CM5] a key estimate for embedded stable annuli which is the main result of this paper (see Theorem 0.3 below). This estimate asserts that such an annulus is a graph away from its boundary if it has only one interior boundary component and if this component lies in a small (extrinsic) ball.

Planar domains arise when one studies convergence of embedded minimal surfaces of a fixed genus in a fixed 3-manifold. This is due to the next theorem which loosely speaking asserts that any sequence of embedded minimal surfaces of fixed genus has a subsequence which are uniformly planar domains away from finitely many points. (In fact, this describes only “1)” and “2)” of Theorem 0.1. Case “3)” is self explanatory and “4)” very roughly corresponds to whether the surface locally “looks like” the genus one helicoid, cf. [HoKrWe], or has “more than one end.”)

Given a surface  $\Sigma$  with boundary  $\partial\Sigma$ , the *genus* of  $\Sigma$  ( $\text{gen}(\Sigma)$ ) is the genus of the closed surface  $\hat{\Sigma}$  obtained by adding a disk to each boundary circle. The genus of a union of disjoint surfaces is the sum of the genera. Therefore, a surface with boundary has nonnegative genus; the genus is zero if and only if it is a planar domain. For example, the disk and the annulus are both genus zero; on the other hand, a closed surface of genus  $g$  with  $k$  disks removed has genus  $g$ .

In the next theorem,  $M^3$  will be a closed 3-manifold and  $\Sigma_i^2$  a sequence of closed embedded oriented minimal surfaces in  $M$  with fixed genus  $g$ .

**Theorem 0.1.** There exist  $x_1, \dots, x_m \in M$  with  $m \leq g$  and a subsequence  $\Sigma_j$  so:

- 1). For  $x \in M \setminus \{x_1, \dots, x_m\}$ , there are  $j_x, r_x > 0$  so  $\text{gen}(B_{r_x}(x) \cap \Sigma_j) = 0$  for  $j > j_x$ .
- 2). For each  $x_k$ , there are  $\ell_k, r_k > 0, r_k > r_{k,j} \rightarrow 0$  so for all  $j$  there are components  $\{\Sigma_{k,j}^\ell\}_{\ell \leq \ell_k}$  of  $B_{r_k}(x_k) \cap \Sigma_j$  with  $\text{gen}(B_{r_k}(x_k) \cap \Sigma_j) = \sum_{\ell \leq \ell_k} \text{gen}(\Sigma_{k,j}^\ell) \leq g$  and  $\text{gen}(\Sigma_{k,j}^\ell) = \text{gen}(B_{r_{k,j}}(x_k) \cap \Sigma_{k,j}^\ell)$  for  $\ell \leq \ell_k$ .
- 3). For every  $k, \ell, j$ , there is only one component  $\tilde{\Sigma}_{k,j}^\ell$  of  $B_{r_{k,j}}(x_k) \cap \Sigma_{k,j}^\ell$  with genus  $> 0$ .
- 4). For each  $k, \ell$ , either  $\partial\Sigma_{k,j}^\ell$  is connected or a component of  $\partial\tilde{\Sigma}_{k,j}^\ell$  separates two components of  $\partial\Sigma_{k,j}^\ell$ .

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To explain why the next two theorems are crucial for what we call “the pair of pants decomposition” of embedded minimal planar domains recall the following prime examples of such domains: Minimal graphs (over disks), a helicoid, a catenoid or one of the Riemann examples. (Note that the first two are topologically disks and the others are disks with one or more subdisks removed.) Let us describe the non simply connected examples in a little more detail. The catenoid is the (topological) annulus

$$(\cosh s \cos t, \cosh s \sin t, s) \tag{0.2}$$

where  $s, t \in \mathbf{R}$ . To describe the Riemann examples, think of a catenoid as roughly being obtained by connecting two parallel planes by a neck. Loosely speaking, the Riemann examples are given by connecting (infinitely many) parallel planes by necks; each adjacent pair of planes is connected by exactly one neck. In addition, all of the necks are lined up along an axis and the separation between each pair of adjacent ends is constant (in fact the surfaces are periodic). Locally, one can imagine connecting  $\ell - 1$  planes by  $\ell - 2$  necks and add half of a catenoid to each of the two outermost planes, possibly with some restriction on how the necks line up and on the separation of the planes; see [FrMe], [Ka], [LoRo].

To illustrate how Theorem 0.3 below will be used in [CM6] where we give the actual “pair of pants decomposition” observe that the catenoid can be decomposed into two minimal annuli each with one exterior convex boundary and one interior boundary which is a short simple closed geodesic. (See also [CM9] for the “pair of pants decomposition” in the special case of annuli.) In the case of the Riemann examples, there will be a number of “pair of pants”, that is, topological disks with two subdisks removed. Metrically these “pair of pants” have one convex outer boundary and two interior boundaries each of which is a simple closed geodesic. Note also that this decomposition can be made by putting in minimal graphical annuli in the complement of the domains (in  $\mathbf{R}^3$ ) which separate each of the pieces; cf. Corollary 0.4 below. Moreover, after the decomposition is made then every intersection of one of the “pair of pants” with an extrinsic ball away from the interior boundaries is simply connected and hence the results of [CM3]–[CM5] apply there.

The next theorem is a kind of effective removable singularity theorem for embedded stable minimal surfaces with small interior boundaries. It asserts that embedded stable minimal surfaces with small interior boundaries are graphical away from the boundary. Here small means contained in a small ball in  $\mathbf{R}^3$  (and not that the interior boundary has small length). This distinction is important; in particular if one had a bound for the area of a tubular neighborhood of the interior boundary, then Theorem 0.3 would follow easily; see Corollary II.1.34 and cf. [Fi].

**Theorem 0.3.** Given  $\tau > 0$ , there exists  $C_1 > 1$ , so if  $\Gamma \subset B_R \subset \mathbf{R}^3$  is an embedded stable minimal annulus with  $\partial\Gamma \subset \partial B_R \cup B_{r_0/4}$  (for  $C_1^2 r_0 < R$ ) and  $B_{r_0} \cap \partial\Gamma$  is connected, then each component of  $B_{R/C_1} \cap \Gamma \setminus B_{C_1 r_0}$  is a graph with gradient  $\leq \tau$ .

Many of the results of this paper will involve either graphs or multi-valued graphs. Graphs will always be assumed to be single-valued over a domain in the plane (as is the case in Theorem 0.3).

For simplicity,  $\Gamma$  in Theorem 0.3 is assumed to be an annulus; see [CM6] for the slight additional arguments needed when  $\Gamma$  is a general planar domain. (Once we see in [CM6] that Theorem 0.3 holds for general planar domains then the corresponding generalization of Corollary 0.4 follows.)

Combining Theorem 0.3 with the solution of a Plateau problem of Meeks-Yau (proven initially for convex domains in theorem 5 of [MeYa1] and extended to mean convex domains in [MeYa2]), we get (the result of Meeks-Yau gives the existence of  $\Gamma$  below):

**Corollary 0.4.** Given  $\tau > 0$ , there exists  $C_1 > 1$ , so: Let  $\Sigma \subset B_R \subset \mathbf{R}^3$ ,  $\partial\Sigma \subset \partial B_R$  be an embedded minimal surface with  $\text{gen}(\Sigma) = \text{gen}(B_{r_1} \cap \Sigma)$  and let  $\Omega$  be a component of  $B_R \setminus \Sigma$ . If  $\gamma \subset B_{r_0} \cap \Sigma \setminus B_{r_1}$  is noncontractible and homologous in  $\Sigma \setminus B_{r_1}$  to a component of  $\partial\Sigma$  and  $r_0 > r_1$ , then a component  $\hat{\Sigma}$  of  $\Sigma \setminus \gamma$  is an annulus and there is a stable embedded minimal annulus  $\Gamma \subset \Omega$  with  $\partial\Gamma = \partial\hat{\Sigma}$ . Moreover, each component of  $(B_{R/C_1} \setminus B_{C_1 r_0}) \cap \Gamma$  is a graph with gradient  $\leq \tau$ .

Stability of  $\Gamma$  in Theorem 0.3 is used in two ways: To get a pointwise curvature bound on  $\Gamma$  and to show that certain sectors have small curvature. In section 2 of [CM4], we showed that a pointwise curvature bound allows us to decompose an embedded minimal surface into a set of bounded area and a collection of (almost stable) sectors with small curvature. Using this, the proof of Theorem 0.3 will also give (if  $0 \in \Sigma$ , then  $\Sigma_{0,t}$  denotes the component of  $B_t \cap \Sigma$  containing 0):

**Theorem 0.5.** Given  $C$ , there exist  $C_2, C_3 > 1$ , so: Let  $0 \in \Sigma \subset B_R \subset \mathbf{R}^3$  be an embedded minimal surface with connected  $\partial\Sigma \subset \partial B_R$ . If  $\text{gen}(\Sigma_{0,r_0}) = \text{gen}(\Sigma)$ ,  $r_0 \leq R/C_2$ , and

$$\sup_{\Sigma \setminus B_{r_0}} |x|^2 |A|^2(x) \leq C, \tag{0.6}$$

then  $\text{Area}(\Sigma_{0,r_0}) \leq C_3 r_0^2$ .

In [CM5] a strengthening of Theorem 0.5 (this strengthening is Theorem III.3.1 below) will be used to show that for limits of a degenerating sequence of embedded minimal disks points where the curvatures blow up are not isolated. This will eventually give (theorem 0.1 of [CM5]) that for a subsequence such points form a Lipschitz curve which is infinite in two directions and transversal to the limit leaves; cf. with the example given by a sequence of rescaled helicoids.

To describe a neighborhood of each of the finitely many points, coming from Theorem 0.1, where the genus concentrates (specifically to describe when there is one component  $\tilde{\Sigma}_{k,j}^\ell$  of genus  $> 0$  in “3.” of Theorem 0.1), we will need in [CM6]:

**Corollary 0.7.** Given  $C, g$ , there exist  $C_4, C_5$  so: Let  $0 \in \Sigma \subset B_R \subset \mathbf{R}^3$  be an embedded minimal surface with connected  $\partial\Sigma \subset \partial B_R$ ,  $r_0 < R/C_4$ , and  $\text{gen}(\Sigma_{0,r_0}) = \text{gen}(\Sigma) \leq g$ . If

$$\sup_{\Sigma \setminus B_{r_0}} |x|^2 |A|^2(x) \leq C, \tag{0.8}$$

then  $\Sigma$  is a disk and  $\Sigma_{0,R/C_5}$  is a graph with gradient  $\leq 1$ .

This corollary follows directly from Theorem 0.5 and theorem 1.22 of [CM4]. Namely, note first that for  $r_0 \leq s \leq R$ , it follows from the maximum principle (since  $\Sigma$  is minimal) and Corollary I.0.11 that  $\partial\Sigma_{0,s}$  is connected and  $\Sigma \setminus \Sigma_{0,s}$  is an annulus. Second, Theorem 0.5 bounds  $\text{Area}(\Sigma_{0,R/C_2})$  and theorem 1.22 of [CM4] then gives the corollary.

Theorems 0.3, 0.5 and Corollary 0.7 are local and are for simplicity stated and proven only in  $\mathbf{R}^3$  although they can with only very minor changes easily be seen to hold for minimal planar domains in a sufficiently small ball in any given fixed Riemannian 3-manifold.

Throughout  $\Sigma$ ,  $\Gamma \subset M^3$  will denote complete minimal surfaces possibly with boundary, sectional curvatures  $K_\Sigma$ ,  $K_\Gamma$ , and second fundamental forms  $A_\Sigma$ ,  $A_\Gamma$ .  $\Gamma$  will be assumed to be stable and have trivial normal bundle. Given  $x \in M$ ,  $B_s(x)$  will be the extrinsic geodesic ball with radius  $s$  and center  $x$ . Likewise, if  $x \in \Sigma$ , then  $\mathcal{B}_s(x)$  is the intrinsic ball in  $\Sigma$ . Given  $S \subset \Sigma$  and  $t > 0$ , let  $\mathcal{T}_t(S, \Sigma) \subset \Sigma$  be the intrinsic tubular neighborhood of  $S$  in  $\Sigma$  with radius  $t$  and set  $\mathcal{T}_{s,t}(S, \Sigma) = \mathcal{T}_t(S, \Sigma) \setminus \mathcal{T}_s(S, \Sigma)$ . Unless explicitly stated otherwise, all geodesics will be parametrized by arclength.

We will often consider the intersections of various curves and surfaces with extrinsic balls. We will always assume that these intersections are transverse since this can anyway be achieved by an arbitrarily small perturbation of the radius.

## Part I. Topological decomposition of surfaces

In this part we will first collect some simple facts and results about planar domains and domains that are planar outside a small ball. These results will then be used to show Theorem 0.1. First we have:

**Lemma I.0.9.** Let  $\Sigma$  be a closed oriented surface (i.e.,  $\partial\Sigma = \emptyset$ ) with genus  $g$ . There are transverse simple closed curves  $\eta_1, \dots, \eta_{2g} \subset \Sigma$  so that for  $i < j$

$$\#\{p \mid p \in \eta_i \cap \eta_j\} = \delta_{i+g,j}. \quad (\text{I.0.10})$$

Furthermore, for any such  $\{\eta_i\}$ , if  $\eta \subset \Sigma \setminus \cup_i \eta_i$  is a closed curve, then  $\eta$  divides  $\Sigma$ .

Recall that if  $\partial\Sigma \neq \emptyset$ , then  $\hat{\Sigma}$  is the surface obtained by replacing each circle in  $\partial\Sigma$  with a disk. Note that a closed curve  $\eta \subset \Sigma$  divides  $\Sigma$  if and only if  $\eta$  is homologically trivial in  $\hat{\Sigma}$ .

**Corollary I.0.11.** If  $\Sigma_1 \subset \Sigma$  and  $\text{gen}(\Sigma_1) = \text{gen}(\Sigma)$ , then each simple closed curve  $\eta \subset \Sigma \setminus \Sigma_1$  divides  $\Sigma$ .

*Proof.* Since  $\Sigma_1$  has genus  $g = \text{gen}(\Sigma)$ , Lemma I.0.9 gives transverse simple closed curves  $\eta_1, \dots, \eta_{2g} \subset \Sigma_1$  satisfying (I.0.10). However, since  $\eta$  does not intersect any of the  $\eta_i$ 's, Lemma I.0.9 implies that  $\eta$  divides  $\Sigma$ .  $\square$

**Corollary I.0.12.** If  $\Sigma$  has a decomposition  $\Sigma = \cup_{\beta=1}^{\ell} \Sigma_\beta$  where the union is taken over the boundaries and each  $\Sigma_\beta$  is a surface with boundary consisting of a number of disjoint circles, then

$$\sum_{\beta=1}^{\ell} \text{gen}(\Sigma_\beta) \leq \text{gen}(\Sigma). \quad (\text{I.0.13})$$

*Proof.* Set  $g_\beta = \text{gen}(\Sigma_\beta)$ . Lemma I.0.9, gives transverse simple closed curves  $\eta_1^\beta, \dots, \eta_{2g_\beta}^\beta \subset \Sigma_\beta$  satisfying (I.0.10). Since  $\Sigma_{\beta_1} \cap \Sigma_{\beta_2} = \emptyset$  for  $\beta_1 \neq \beta_2$ , this implies that the rank of the intersection form on the first homology (mod 2) of  $\hat{\Sigma}$  is  $\geq 2 \sum_{\beta=1}^{\ell} g_\beta$ . In particular, we get (I.0.13).  $\square$

In the next lemma,  $M^3$  will be a closed 3-manifold and  $\Sigma_i^2$  a sequence of closed embedded oriented minimal surfaces in  $M$  with fixed genus  $g$ .

**Lemma I.0.14.** There exist  $x_1, \dots, x_m \in M$  with  $m \leq g$  and a subsequence  $\Sigma_j$  so: For  $x \in M \setminus \{x_1, \dots, x_m\}$ , there exist  $j_x, r_x > 0$  so  $\text{gen}(B_{r_x}(x) \cap \Sigma_j) = 0$  for  $j > j_x$ . For each  $x_k$ , there exist  $R_k, g_k > 0, R_k > R_{k,j} \rightarrow 0$  so  $\text{gen}(B_{R_k}(x_k) \cap \Sigma_j) = g_k = \text{gen}(B_{R_{k,j}}(x_k) \cap \Sigma_j)$  for all  $j$  and  $\sum_{k=1}^m g_k \leq g$ .

*Proof.* Suppose that for some  $x_1 \in M$  and any  $R_1 > 0$  we have  $\text{gen}(B_{R_1}(x_1) \cap \Sigma_i) = g_{1,i} > 0$  for infinitely many  $i$ 's. By Corollary I.0.12,  $g_{1,i} \leq g$  and so there is a subsequence  $\Sigma_j$  and a sequence  $R_{1,j} \rightarrow 0$  so that for all  $j$

$$\text{gen}(B_{R_{1,j}}(x_1) \cap \Sigma_j) = g_1 > 0. \quad (\text{I.0.15})$$

By repeating this construction, we can suppose that there are disjoint points  $x_1, \dots, x_m \in M$  and  $R_{k,j} > 0$  so that for any  $k$  we have  $R_{k,j} \rightarrow 0$  and  $\text{gen}(B_{R_{k,j}}(x_k) \cap \Sigma_j) = g_k > 0$ .

However, Corollary I.0.12 implies that for  $j$  sufficiently large

$$0 \leq \text{gen}(\Sigma_j \setminus \cup_k B_{R_{k,j}}(x_k)) \leq \text{gen}(\Sigma_j) - \sum_{k=1}^m \text{gen}(B_{R_{k,j}}(x_k) \cap \Sigma_j) \leq g - \sum_{k=1}^m g_k. \quad (\text{I.0.16})$$

In particular,  $\sum_{k=1}^m g_k \leq g$  and we can therefore assume that  $\sum_{k=1}^m g_k$  is maximal. This has two consequences. First, given  $x \in M \setminus \{x_1, \dots, x_m\}$ , there exist  $r_x > 0$  and  $j_x$  so that  $\text{gen}(B_{r_x}(x) \cap \Sigma_j) = 0$  for  $j > j_x$ . Second, for each  $x_k$ , there exist  $R_k > 0$  and  $j_k$  so  $\text{gen}(B_{R_k}(x_k) \cap \Sigma_j) = g_k$  for  $j > j_k$ . The lemma now follows easily.  $\square$

By Corollary I.0.12, each  $R_k, R_{k,j}$  from Lemma I.0.14 can (after going to a further subsequence) be replaced by any  $R'_k, R'_{k,j}$  with  $R'_k \leq R_k$  and  $R'_{k,j} \geq R_{k,j}$ . Similarly, each  $r_x$  can be replaced by any  $r'_x \leq r_x$ . This will be used freely in the proof of Theorem 0.1 below.

*Proof.* (of Theorem 0.1). Let  $x_k, g_k, R_k, R_{k,j}$  and  $r_x$  be from Lemma I.0.14. We can assume that each  $R_k > 0$  is sufficiently small so that  $B_{R_k}(x_k)$  is essentially Euclidean (e.g.,  $R_k < \min\{i_0/4, \pi/(4k^{1/2})\}$ ). “1.” follows directly from Lemma I.0.14.

For each  $x_k$ , we can assume that there are  $\ell_k$  and  $n_{\ell,k}$  so:  $B_{R_k}(x_k) \cap \Sigma_j$  has components  $\{\Sigma_{k,j}^\ell\}_{1 \leq \ell \leq \ell_k}$  with genus  $> 0$  and  $B_{R_{k,j}}(x_k) \cap \Sigma_{k,j}^\ell$  has  $n_{\ell,k}$  components with genus  $> 0$ . We will use repeatedly that, by “1.” and Corollary I.0.12:  $n_{\ell,k}$  is nonincreasing if either  $R_{k,j}$  increases or  $R_k$  decreases. For each  $\ell, k$  with  $n_{\ell,k} > 1$ , set

$$\rho_{k,j}^\ell = \inf\{\rho > R_{k,j} \mid \#\{\text{components of } B_\rho(x_k) \cap \Sigma_{k,j}^\ell\} < n_{\ell,k}\}. \quad (\text{I.0.17})$$

There are two cases. If  $\liminf_{j \rightarrow \infty} \rho_{k,j}^\ell = 0$ , then choose a subsequence  $\Sigma_j$  with  $\rho_{k,j}^\ell \rightarrow 0$ ;  $n_{\ell,k}$  decreases if we replace  $R_{k,j}$  with any  $R'_{k,j} > \rho_{k,j}^\ell$ . Otherwise, set  $2\rho_k^\ell = \liminf_{j \rightarrow \infty} \rho_{k,j}^\ell > 0$  and choose a subsequence  $\Sigma_j$  so  $\rho_{k,j}^\ell < \rho_k^\ell$ ;  $\ell_k$  increases if we replace  $R_k$  with any  $R'_k \leq \rho_k^\ell$ . In either case,  $\sum_{\ell,k} (n_{\ell,k} - 1)$  decreases. Since  $\sum_{\ell,k} n_{\ell,k} \leq g$  (by Corollary I.0.12), repeating this  $\leq g$  times gives  $0 < R'_k \leq R_k, R_{k,j} \leq R'_{k,j} \rightarrow 0$  and a subsequence so only one component  $\tilde{\Sigma}_{k,j}^\ell$  of  $B_{R'_{k,j}}(x_k) \cap \Sigma_{k,j}^\ell$  has genus  $> 0$  (i.e., each new  $n_{\ell,k} = 1$ ). By Corollary I.0.12 (and “1.”) and the remarks before the proof, “1.”, “2.”, and “3.” now hold for any  $r_k \leq R'_k$  and  $R'_{k,j} \leq r_{k,j} \rightarrow 0$ .

Suppose that for some  $k, \ell$  there exists  $j_{k,\ell}$  so  $\partial \Sigma_{k,j}^\ell$  has at least two components for all  $j > j_{k,\ell}$ . For  $R'_{k,j} \leq t \leq R'_k$ , let  $\Sigma_{k,j}^\ell(t)$  be the component of  $B_t(x_k) \cap \Sigma$  containing  $\tilde{\Sigma}_{k,j}^\ell$ . Set

$$r_{k,j}^\ell = \inf\{t > R_{k,j} \mid \#\{\text{components of } \partial \Sigma_{k,j}^\ell(t)\} > 1\}. \quad (\text{I.0.18})$$

There are two cases. If  $\liminf_{j \rightarrow \infty} r_{k,j}^\ell = 0$ , then choose a subsequence  $\Sigma_j$  with  $r_{k,j}^\ell \rightarrow 0$ . By the maximum principle (since  $\Sigma$  is minimal) and Corollary I.0.11, a component of (the new)  $\partial \Sigma_{k,j}^\ell$  separates two components of  $\partial \Sigma_{k,j}^\ell$  for any  $r_{k,j} \rightarrow 0$  with  $r_{k,j} > r_{k,j}^\ell$ . On the other hand, if  $\liminf_{j \rightarrow \infty} r_{k,j}^\ell = 2r_k^\ell > 0$ , then choose a subsequence so (the new)  $\partial \Sigma_{k,j}^\ell$  is connected for any  $r_k \leq r_k^\ell$ . After repeating this  $\leq g$  times (each time either increasing  $R'_{k,j}$  or decreasing  $R'_k$ ), “4)” also holds.  $\square$

In [CM6] we will need the following (here, and elsewhere, if  $0 \in \Sigma \subset \mathbf{R}^3$ , then  $\Sigma_{0,t}$  denotes the component of  $B_t \cap \Sigma$  containing 0):

**Proposition I.0.19.** Let  $0 \in \Sigma_i \subset B_{S_i} \subset \mathbf{R}^3$  with  $\partial \Sigma_i \subset \partial B_{S_i}$  be a sequence of embedded minimal surfaces with genus  $\leq g < \infty$  and  $S_i \rightarrow \infty$ . After going to a subsequence,  $\Sigma_j$ , and possibly replacing  $S_j$  by  $R_j$  and  $\Sigma_j$  by  $\Sigma_{0,j,R_j}$  where  $R_0 \leq R_j \leq S_j$  and  $R_j \rightarrow \infty$ , then  $\text{gen}(\Sigma_{j,0,R_0}) = \text{gen}(\Sigma_j) \leq g$  and either (a) or (b) holds:

- (a)  $\partial \Sigma_{j,0,t}$  is connected for all  $R_0 \leq t \leq R_j$ .
- (b)  $\partial \Sigma_{j,0,R_0}$  is disconnected.

*Proof.* We will first show that there exists  $R_0 > 0$ , a subsequence  $\Sigma_j$ , and a sequence  $R_j \rightarrow \infty$  with  $R \leq R_j \leq S_j$ , such that (after replacing  $\Sigma_j$  by  $\Sigma_{j,0,R_j}$ )  $\text{gen}(\Sigma_{j,0,R_0}) = \text{gen}(\Sigma_j) \leq g$ . Suppose not; it follows easily from the monotonicity of the genus (i.e., Corollary I.0.12) that there exists a subsequence  $\Sigma_j$  and a sequence  $G_k \rightarrow \infty$  such that for all  $k$  there exists a  $j_k$  so for  $j \geq j_k$

$$g \geq \text{gen}(\Sigma_{j,0,G_{k+1}}) > \text{gen}(\Sigma_{j,0,G_k}), \quad (\text{I.0.20})$$

which is a contradiction.

For each  $j$ , let  $R_{0,j}$  be the infimum of  $R$  with  $R_0 \leq R \leq R_j$  where  $\partial \Sigma_{j,0,R}$  is disconnected; set  $R_{0,j} = R_j$  if no such exists. Either  $\liminf R_{0,j} < \infty$ , in which case, after going to a subsequence and replacing  $R_0$  by  $\liminf R_{0,j} + 1$ , we are in (b) by the maximum principle. Or, if  $\liminf R_{0,j} = \infty$ , then we are in (a) after replacing  $R_j$  by  $R_{0,j}$ .  $\square$

## Part II. Estimates for stable minimal surfaces with small interior boundaries

In this part we prove Theorem 0.3.

### II.1. LONG STABLE SECTORS CONTAIN MULTI-VALUED GRAPHS

In [CM3], [CM4] we gave estimates for stable sectors. A stable sector in the sense of [CM3], [CM4] is a stable subset of a minimal surface given as half of a normal tubular neighborhood (in the surface) of a strictly convex curve (for instance, a curve lying in the boundary of an intrinsic ball). In this section we give similar estimates for half of normal tubular neighborhoods of curves lying in the intersection of the surface and the boundary of an extrinsic ball. These domains arise naturally in our main result and are unfortunately somewhat more complicated to deal with due to the lack of convexity of the curves.

In this section, the surfaces  $\Sigma$  and  $\Gamma$  will be planar domains and, hence, simple closed curves will divide the surface into two planar (sub)domains.

We will need some notation for multi-valued graphs. Let  $\mathcal{P}$  be the universal cover of the punctured plane  $\mathbf{C} \setminus \{0\}$  with global (polar) coordinates  $(\rho, \theta)$  and set  $S_{r,s}^{\theta_1, \theta_2} = \{r \leq \rho \leq$

$s, \theta_1 \leq \theta \leq \theta_2$ . An  $N$ -valued graph  $\Sigma$  of a function  $u$  over the annulus  $D_s \setminus D_r$  is a (single-valued) graph (of  $u$ ) over  $S_{r,s}^{-N\pi, N\pi}$  ( $\Sigma_{r,s}^{\theta_1, \theta_2}$  will denote the subgraph of  $\Sigma$  over  $S_{r,s}^{\theta_1, \theta_2}$ ). The separation  $w(\rho, \theta)$  between consecutive sheets is

$$w(\rho, \theta) = u(\rho, \theta + 2\pi) - u(\rho, \theta). \quad (\text{II.1.1})$$

The main result of the next two sections is ( $\Gamma_1(\partial)$  is the component of  $B_1 \cap \Gamma$  containing  $B_1 \cap \partial\Gamma$ ):

**Theorem II.1.2.** Given  $N, \tau > 0$ , there exist  $\omega > 1, d_0$  so: Let  $\Gamma$  be a stable embedded minimal annulus with  $\partial\Gamma \subset B_{1/4} \cup \partial B_R$ ,  $B_{1/4} \cap \partial\Gamma$  connected, and  $R > \omega^2$ . Given  $z_1 \in \partial B_1 \cap \partial\Gamma_1(\partial)$ , (after a rotation of  $\mathbf{R}^3$ ) either (1) or (2) below holds:

- (1) Each component of  $B_{R/\omega} \cap \Gamma \setminus B_\omega$  is a graph with gradient  $\leq \tau$ .
- (2)  $\Gamma$  contains a graph  $\Gamma_{\omega, R/\omega}^{-N\pi, N\pi}$  with gradient  $\leq \tau$  and  $\text{dist}_{\Gamma \setminus \Gamma_1(\partial)}(z_1, \Gamma_{\omega, \omega}^{0,0}) < d_0$ .

Note that if  $\Gamma$  is as in Theorem II.1.2 and one component of  $B_{R/\omega} \cap \Gamma \setminus B_\omega$  contains a graph over  $D_{R/(2\omega)} \setminus D_{2\omega}$  with gradient  $\leq 1$ , then every component of  $B_{R/(C\omega)} \cap \Gamma \setminus B_{C\omega}$  is a graph for some  $C > 1$ . Namely, embeddedness and the gradient estimate (which applies because of stability) would force any nongraphical component to spiral indefinitely, contradicting that  $\Gamma$  is compact. Thus it is enough to find one component that is a graph. This we be used below.

We will eventually show in Section II.3 that (2) in Theorem II.1.2 does not happen; thus every component is a (single-valued) graph. This will easily give Theorem 0.3.

Throughout this section (except in Corollary II.1.34),  $\Sigma \subset \mathbf{R}^3$  will be an embedded minimal planar domain (if the domain is stable, then we use  $\Gamma$  instead of  $\Sigma$ ),  $\Sigma_0 \subset \Sigma$  a subdomain and  $\gamma_1, \gamma_2, \sigma_1 \subset \partial\Sigma_0$  curves ( $\gamma_1, \gamma_2$  geodesics) so  $\gamma_1 \cup \gamma_2 \cup \sigma_1$  is a simple curve and  $\gamma_i(0) \in \sigma_1$ . (By a geodesic we will mean a curve with zero geodesic curvature. This definition of geodesic is needed when the curve intersect the boundary of the surface.) Below we will sometimes require one or more of the following properties:

- (A)  $\text{dist}_\Sigma(\gamma_i(t), \sigma_1) \geq t - C_0$  for  $0 \leq t \leq \text{Length}(\gamma_i)$ .
- (B)  $\partial_{\mathbf{n}}|x| \geq 0$  along  $\sigma_1$  (where  $\mathbf{n}$  is the inward normal to  $\partial\Sigma_0$ ).
- (C)  $\gamma_1 \perp \sigma_1, \gamma_2 \perp \sigma_1$  (i.e., angle  $\pi/2$ ).
- (D)  $\text{dist}_{\Sigma_0}(\sigma_1, \partial\Sigma_0 \setminus (\sigma_1 \cup \gamma_1 \cup \gamma_2)) \geq \ell$  (thus  $\ell \leq \text{Length}(\gamma_i)$ ).

Note that if  $\sigma_1 \subset \partial B_1$  (and  $\Sigma_0$  is leaving  $B_1$  along  $\sigma_1$ ), then (B) is automatically satisfied.

The main component of the proof of Theorem II.1.2 is Proposition II.1.20 below which shows that certain stable sectors have subsectors with small total curvature. To show that we will use an argument in the spirit of [CM2], [CM4] to get good curvature estimates for our nonstandard stable domains. As in [CM2], [CM4], to estimate the total curvature we show first an area bound. That is (here  $k_g$  is the geodesic curvature of  $\sigma_1$ ):

**Lemma II.1.3.** Let  $\Gamma_0 = \Gamma \subset \mathbf{R}^3$  be stable and satisfy (A) for  $C_0 = 0$ , (C), (D). If  $0 \leq \chi \leq 1$  is a function on  $\Gamma_0$  which vanishes on each  $\gamma_i$ , then for  $1 < R < \ell$

$$\begin{aligned} \text{Area}(\mathcal{T}_R(\sigma_1, \Gamma_0)) &\leq C R^2 \int_{\sigma_1} |k_g| + C R \text{Length}(\sigma_1) \\ &+ C R^2 \left( \int_{\mathcal{T}_1(\sigma_1, \Gamma_0)} (1 + |A|^2) + \int_{\mathcal{T}_R(\sigma_1, \Gamma_0)} |\nabla \chi|^2 + \int_{\mathcal{T}_R(\sigma_1, \Gamma_0) \cap \{\chi < 1\}} |A|^2 \right). \end{aligned} \quad (\text{II.1.4})$$

*Proof.* Set  $\mathcal{T}_{s,t} = \mathcal{T}_{s,t}(\sigma_1, \Gamma_0)$  and  $r = \text{dist}_\Gamma(\sigma_1, \cdot)$ . Define a (radial) cut-off function  $\phi$  by

$$\phi = \begin{cases} r & \text{on } \mathcal{T}_1, \\ (R-r)/(R-1) & \text{on } \mathcal{T}_{1,R}, \\ 0 & \text{otherwise.} \end{cases} \quad (\text{II.1.5})$$

By the stability inequality applied to  $\phi\chi$  and using the inequality,  $2ab \leq a^2 + b^2$ ,

$$\begin{aligned} \int_{\mathcal{T}_{1,R}} |A|^2 [(R-r)/(R-1)]^2 &\leq \int |A|^2 \phi^2 \leq 2 \int |\nabla \phi|^2 + 2 \int_{\mathcal{T}_R} |\nabla \chi|^2 + \int_{\mathcal{T}_R \cap \{\chi < 1\}} |A|^2 \\ &\leq 2 \text{Area}(\mathcal{T}_1) + 2(R-1)^{-2} \text{Area}(\mathcal{T}_{1,R}) + 2 \int_{\mathcal{T}_R} |\nabla \chi|^2 + \int_{\mathcal{T}_R \cap \{\chi < 1\}} |A|^2. \end{aligned} \quad (\text{II.1.6})$$

Set  $K(s) = \int_{\mathcal{T}_{1,s}} |A|^2$ . By the coarea formula and integrating (II.1.6) by parts twice, we get

$$\begin{aligned} 2(R-1)^{-2} \int_1^R \int_1^t K(s) ds dt &\leq 2/(R-1) \int_1^R K(s)(R-s)/(R-1) ds \\ &\leq \int_1^R K'(s) ((R-s)/(R-1))^2 ds \\ &\leq 2 \text{Area}(\mathcal{T}_1) + 2(R-1)^{-2} \text{Area}(\mathcal{T}_{1,R}) + 2 \int_{\mathcal{T}_R} |\nabla \chi|^2 + \int_{\{\chi < 1\}} |A|^2. \end{aligned} \quad (\text{II.1.7})$$

Given  $y \in \sigma_1$ , let  $\gamma_y : [0, r_y] \rightarrow \Gamma$  be the (inward from  $\partial\Gamma$ ) normal geodesic up to the cut-locus of  $\sigma_1$  (so  $\text{dist}_\Gamma(\sigma_1, \gamma_y(r_y)) = r_y$ ) and  $J_y$  the corresponding Jacobi field with  $J_y(0) = 1$  and  $J'_y(0) = k_g(y)$ . Set  $R_y = \min\{r_y, R\}$ . By the Jacobi equation,

$$\int_0^{R_y} J_y(s) ds = R_y^2 k_g(y)/2 + R_y - \int_0^{R_y} \int_0^t \int_0^s K_\Gamma(\gamma_y(\tau)) J_y(\tau) d\tau ds dt. \quad (\text{II.1.8})$$

If  $R_y < R$ , then we extend  $J_y(\tau)$ ,  $K_y(\tau) = K_\Gamma(\gamma_y(\tau))$  to functions  $\tilde{J}_y, \tilde{K}_y$  on  $[0, R]$  by setting  $\tilde{J}_y = J_y, \tilde{K}_y = K_y$  on  $[0, R_y]$  and  $\tilde{J}_y = \tilde{K}_y = 0$  otherwise. If  $R_y = R$ , then we set  $\tilde{J}_y = J_y$  and  $\tilde{K}_y = K_y$ . Since  $K_\Gamma = -|A|^2/2$  (in particular, is  $\leq 0$ ), by (II.1.8)

$$\int_0^{R_y} J_y(s) ds \leq R^2 |k_g(y)|/2 + R - \int_0^R \int_0^t \int_0^s \tilde{K}_y(\tau) \tilde{J}_y(\tau) d\tau ds dt. \quad (\text{II.1.9})$$

Since  $K(s) = -2 \int_{\sigma_1} \int_1^s \tilde{K}_y(\tau) \tilde{J}_y(\tau) d\tau dy$  (this uses (C)), integrating (II.1.9) over  $\sigma_1$  gives

$$\text{Area}(\mathcal{T}_R) \leq \frac{R^2}{2} \int_{\sigma_1} |k_g| + R \text{Length}(\sigma_1) + \int_1^R \int_1^t \frac{K(s)}{2} ds dt + \frac{R^2}{2} \int_{\mathcal{T}_1} |A|^2. \quad (\text{II.1.10})$$

(Here we also used  $\int_0^R \int_0^t f(s) ds dt \leq \int_1^R \int_1^t [f(s) - f(1)] ds dt + R^2 f(1)$  for the nondecreasing function  $f(t) = \int_{\mathcal{T}_t} |A|^2 \geq 0$ .) Combining (II.1.7) and (II.1.10) gives (II.1.4).  $\square$

To apply Lemma II.1.3, we will need to replace a given curve, in a minimal disk, by a curve lying within a fixed tubular neighborhood of it and with length and total geodesic curvature bounded in terms of the area of the tubular neighborhood. This is:

**Lemma II.1.11.** If  $\Sigma \subset \mathbf{R}^3$  is an immersed minimal disk,  $\partial\Sigma = \gamma_1 \cup \gamma_2 \cup \sigma_1 \cup \sigma_2$ , the  $\gamma_i$ 's are geodesics with  $2 \leq \text{Length}(\gamma_i) = \text{dist}_\Sigma(\sigma_2 \cap \gamma_i, \sigma_1)$ ,  $1 \leq \text{dist}_\Sigma(\sigma_1, \sigma_2)$ , then there exists a simple curve  $\check{\sigma}_1 \subset \mathcal{T}_{1/64, 1/4}(\sigma_1)$  connecting  $\gamma_1$  to  $\gamma_2$  and with

$$\text{Length}(\check{\sigma}_1) + \int_{\check{\sigma}_1} |k_g| \leq C_1 (1 + \text{Area}(\mathcal{T}_{1/4}(\sigma_1))). \quad (\text{II.1.12})$$

Moreover, we can choose  $\check{\sigma}_1$  to intersect  $\gamma_i$  orthogonally and so  $\text{Length}(\check{\gamma}_i) = \text{dist}_\Sigma(\sigma_2 \cap \gamma_i, \check{\sigma}_1)$ , where  $\check{\gamma}_i$  denotes the component of  $\gamma_i \setminus \check{\sigma}_1$  which intersects  $\sigma_2$ .

*Proof.* We will do this in three steps. First, we use the coarea formula to find a level set of the distance function with bounded length. Local replacement then gives a broken geodesic with the same length bound and a bound on the number of breaks. Third, we find a simple subcurve and use the Gauss-Bonnet theorem to control the number of breaks.

Set  $r(\cdot) = \text{dist}_\Sigma(\sigma_1, \cdot)$ . By the coarea formula applied to (a regularization of)  $r$ , there exists  $d_0$  between  $1/16$  and  $3/32$  with  $\text{Length}(\{r = d_0\}) \leq 32 \text{Area}(\mathcal{T}_{1/8}(\sigma_1))$  and so that  $\{r = d_0\}$  is transverse. Since  $\{r = d_0\}$  separates  $\sigma_1$  and  $\sigma_2$ , a component  $\tilde{\sigma}$  of  $\{r = d_0\}$  goes from  $\gamma_1$  to  $\gamma_2$ .

Parametrize  $\tilde{\sigma}$  by arclength and let  $0 = t_0 < \dots < t_n = \text{Length}(\tilde{\sigma})$  be a subdivision with  $t_{i+1} - t_i \leq 1/32$  and  $n \leq 32 \text{Length}(\tilde{\sigma}) + 1$ . Since  $\mathcal{B}_{1/32}(y)$  is a disk for all  $y \in \tilde{\sigma}$ , it follows that we can replace  $\tilde{\sigma}$  with a broken geodesic  $\tilde{\sigma}_1$  with breaks at  $\tilde{\sigma}(t_i) = \tilde{\sigma}_1(t_i)$  and which is homotopic to  $\tilde{\sigma}$  in  $\mathcal{T}_{1/32}(\tilde{\sigma})$ . We can assume that  $\tilde{\sigma}_1$  intersects the  $\gamma_i$ 's only at its endpoints.

Let  $[a, b]$  be a maximal interval so that  $\tilde{\sigma}_1|_{[a, b]}$  is simple. We are done if  $\tilde{\sigma}_1|_{[a, b]} = \tilde{\sigma}_1$ . Otherwise,  $\tilde{\sigma}_1|_{[a, b]}$  bounds a disk in  $\Sigma$  and the Gauss-Bonnet theorem implies that  $\tilde{\sigma}_1|_{(a, b)}$  contains a break. Hence, replacing  $\tilde{\sigma}_1$  by  $\tilde{\sigma}_1 \setminus \tilde{\sigma}_1|_{(a, b)}$  gives a subcurve from  $\gamma_1$  to  $\gamma_2$  but does not increase the number of breaks. Repeating this eventually gives a simple subcurve with the same bounds for the length and the number of breaks. Smoothing this at the breaks gives the desired  $\check{\sigma}_1$ .

Finally, since  $\gamma_i$  minimizes distance from  $\gamma_i \cap \sigma_2$  to  $\sigma_1$ , it follows easily by adding segments in  $\gamma_1, \gamma_2$  to  $\check{\sigma}_1$  and then perturbing infinitesimally near  $\gamma_1, \gamma_2$  that we can choose  $\check{\sigma}_1$  to intersect  $\gamma_i$  orthogonally and so each  $\check{\gamma}_i$  minimizes distance back to  $\check{\sigma}_1$ ; this gives at most a bounded contribution to the length and total curvature.  $\square$

We will also need a version of Lemma II.1.11 where  $\sigma$  is a noncontractible curve (cf. lemma 1.21 in [CM4]):

**Lemma II.1.13.** Let  $\Sigma \subset \mathbf{R}^3$  be an immersed minimal planar domain and  $\sigma = B_1 \cap \partial\Sigma$  a simple closed curve with  $\text{dist}_\Sigma(\sigma, \partial\Sigma \setminus \sigma) > 1$ . There exists a simple noncontractible curve  $\check{\sigma} \subset \mathcal{T}_{1/32, 1/4}(\sigma)$  with

$$\text{Length}(\check{\sigma}) + \int_{\check{\sigma}} |k_g| \leq C_1 (1 + \text{Area}(\mathcal{T}_{1/4}(\sigma))). \quad (\text{II.1.14})$$

*Proof.* Following the first two steps of the proof of Lemma II.1.11 (with the obvious modifications), we get a simple closed broken geodesic  $\tilde{\sigma}_1$  which is noncontractible with length and the number of breaks  $\leq C \text{Area}(\mathcal{T}_{1/4}(\sigma))$ .

As in the third step of the proof of Lemma II.1.11, let  $\tilde{\sigma}_1|_{[a, b]}$  be a maximal simple subcurve. It follows that  $\tilde{\sigma}_1|_{[a, b]}$  is closed (and has at most one more break than  $\tilde{\sigma}_1$ ). If  $\tilde{\sigma}_1|_{[a, b]}$  is noncontractible, then we are done. Otherwise, if  $\tilde{\sigma}_1|_{[a, b]}$  bounds a disk, then we apply the

Gauss-Bonnet theorem to see that  $\tilde{\sigma}_1|_{(a,b)}$  contains a break and proceed as in the proof of Lemma II.1.11.  $\square$

In Proposition II.1.20 below, we will also need a lower bound for the area growth of tubular neighborhoods of a curve. To get such a bound, we need that the curve is not completely “crumpled up.” This will follow by requiring that  $(t + C_0)(t + 1) \leq \delta \text{Area}(\mathcal{T}_1(\sigma_1))$ .

**Lemma II.1.15.** Let  $\Sigma_0 = \Sigma$  satisfy (A), (B) and (D). If  $\sigma_1 \subset B_1$ ,  $1 \leq s < t \leq \ell$  and  $(t + C_0)(t + 1) \leq \delta \text{Area}(\mathcal{T}_1(\sigma_1))$ , then

$$(t + 1)^{2\delta-2} \text{Area}(\mathcal{T}_t(\sigma_1)) \geq (s + 1)^{2\delta-2} \text{Area}(\mathcal{T}_s(\sigma_1)). \quad (\text{II.1.16})$$

*Proof.* Set  $\mathcal{T}_t = \mathcal{T}_t(\sigma_1)$  and  $L(s) = \int_{\partial\mathcal{T}_s \setminus \partial\Sigma} 1$ . By minimality, Stokes’ theorem, (A), (B) and  $\text{dist}_\Sigma(\sigma_1, x) + 1 \geq |x|$ , we get that

$$4 \text{Area}(\mathcal{T}_s) = \int_{\mathcal{T}_s} \Delta |x|^2 \leq 2(s + 1)L(s) + 4(s + C_0)(s + 1). \quad (\text{II.1.17})$$

By the coarea formula,  $(\text{Area}(\mathcal{T}_s))' = L(s)$  for almost every  $s$ . Hence, for almost every  $s$  with  $\text{dist}_\Sigma(\sigma_1, \sigma_2) \geq s \geq 1$

$$(\log \text{Area}(\mathcal{T}_s))' \geq \frac{2}{s + 1} - \frac{2(s + C_0)}{\text{Area}(\mathcal{T}_s)} \geq \frac{2(1 - \delta)}{s + 1}. \quad (\text{II.1.18})$$

Since  $\text{Area}(\mathcal{T}_s)$  is monotone, a standard argument then gives (II.1.16).  $\square$

**Remark II.1.19.** In the special case of Lemma II.1.15 where  $\Sigma$  is an annulus with  $\partial\Sigma = \sigma_1 \cup \sigma_2$ , i.e., where  $\gamma_i = \emptyset$  and  $\sigma_1, \sigma_2$  are closed, the proof simplifies in an obvious way and  $\delta$  can be chosen to be zero.

We are now ready to apply Lemma II.1.3 and use the logarithmic cut-off trick to show that certain stable sectors have small curvature:

**Proposition II.1.20.** Let  $\Gamma_0 \subset \Gamma \subset \mathbf{R}^3$  satisfy (A) (with  $C_0 = 0$ ), (B), (D), and  $\text{dist}_\Gamma(\Gamma_0, \partial\Gamma) > 1/4$ . Suppose that  $\Gamma$  is stable,  $\omega > 2$ ,  $\ell > R_0 > \omega^2$ , and  $\sigma_1 \subset B_1$ . If  $\Gamma_0$  is a disk and  $4R_0^2(R_0 + 1) \leq \text{Area}(\mathcal{T}_1(\sigma_1, \Gamma_0))$ , then for  $\omega^2 \leq t \leq R_0$

$$\text{Area}(\mathcal{T}_2(\sigma_1, \Gamma_0)) t^2 / C \leq \text{Area}(\mathcal{T}_{\omega,t}(\sigma_1, \Gamma_0)) \leq C \text{Area}(\mathcal{T}_2(\sigma_1, \Gamma_0)) t^2, \quad (\text{II.1.21})$$

$$\int_{\mathcal{T}_{\omega,R_0/\omega}(\sigma_1, \Gamma_0)} |A|^2 \leq C R_0 + \frac{C}{\log \omega} \text{Area}(\mathcal{T}_2(\sigma_1, \Gamma_0)). \quad (\text{II.1.22})$$

*Proof.* Define a function  $\chi$  on  $\Gamma_0$  by

$$\chi = \begin{cases} 2 \text{dist}_\Gamma(\gamma_1 \cup \gamma_2, \cdot) & \text{on } \mathcal{T}_{1/2}(\gamma_1 \cup \gamma_2), \\ 1 & \text{otherwise} . \end{cases} \quad (\text{II.1.23})$$

We will use  $\chi$  to cut-off on the sides  $\gamma_1, \gamma_2$ . Using [Sc], [CM2], and  $\text{dist}_\Gamma(\Gamma_0, \partial\Gamma) > 1/4$ ,

$$\int_{\mathcal{T}_2(\sigma_1, \Gamma_0)} (1 + |A|^2) \leq C_1 \text{Area}(\mathcal{T}_2(\sigma_1, \Gamma_0)), \quad (\text{II.1.24})$$

$$2 \int_{\mathcal{T}_{R_0}(\sigma_1, \Gamma_0)} |\nabla \chi|^2 + \int_{\mathcal{T}_{R_0}(\sigma_1, \Gamma_0) \cap \{\chi < 1\}} |A|^2 \leq C_1 R_0 \leq C_1 \text{Area}(\mathcal{T}_1(\sigma_1, \Gamma_0)). \quad (\text{II.1.25})$$

Since  $\sigma_1 \subset \partial\Gamma_0$  satisfies (A) with  $C_0 = 0$  and (D), Lemma II.1.11 gives a simple curve  $\check{\sigma}_1$  (and  $\check{\gamma}_1, \check{\gamma}_2$ ) satisfying (A) with  $C_0 = 0$ , (C), (D), and (II.1.12); let  $\check{\Gamma}_0 \subset \Gamma_0$  be the component of  $\Gamma_0 \setminus \check{\sigma}_1$  containing  $\sigma_2$ . By the triangle inequality,

$$\mathcal{T}_t(\check{\sigma}_1, \Gamma_0) \subset \mathcal{T}_{t+1/4}(\sigma_1, \Gamma_0) \subset \mathcal{T}_{t+1/4}(\check{\sigma}_1, \check{\Gamma}_0) \cup (\Gamma_0 \setminus \check{\Gamma}_0). \quad (\text{II.1.26})$$

Note that  $\Gamma_0 \setminus \check{\Gamma}_0$  is a disk with boundary  $\sigma_1 \cup \check{\sigma}_1 \cup (\gamma_1 \setminus \check{\gamma}_1) \cup (\gamma_2 \setminus \check{\gamma}_2)$ . Hence, by minimality, Stokes' theorem, (B),  $|x| \leq 5/4$  on  $\partial(\Gamma_0 \setminus \check{\Gamma}_0)$ , and (II.1.12),

$$4 \text{Area}(\Gamma_0 \setminus \check{\Gamma}_0) = \int_{\Gamma_0 \setminus \check{\Gamma}_0} \Delta |x|^2 \leq 2 \int_{\check{\sigma}_1 \cup (\gamma_1 \setminus \check{\gamma}_1) \cup (\gamma_2 \setminus \check{\gamma}_2)} |x| \leq C'_1 \text{Area}(\mathcal{T}_1(\sigma_1, \Gamma_0)). \quad (\text{II.1.27})$$

Inserting (II.1.24), (II.1.25) into Lemma II.1.3 applied to  $\check{\sigma}_1$  and using (II.1.12), (II.1.26), (II.1.27) gives for  $2 \leq t \leq R_0$

$$\text{Area}(\mathcal{T}_t(\sigma_1, \Gamma_0)) \leq C_2 \text{Area}(\mathcal{T}_2(\sigma_1, \Gamma_0)) t^2, \quad (\text{II.1.28})$$

giving the second inequality in (II.1.21). Set  $\mathcal{T}_t = \mathcal{T}_t(\sigma_1, \Gamma_0)$  (define  $\mathcal{T}_{s,t}$  similarly) and set  $L(t) = \int_{\partial\mathcal{T}_t \setminus \partial\Gamma_0} 1$ . By (II.1.28), the coarea formula, and integration by parts,

$$\begin{aligned} \int_{R_0/\omega}^{R_0} L(t) t^{-2} dt &= \left[ \text{Area}(\mathcal{T}_{R_0/\omega, t}) t^{-2} \right]_{R_0/\omega}^{R_0} + 2 \int_{R_0/\omega}^{R_0} \text{Area}(\mathcal{T}_{R_0/\omega, t}) t^{-3} dt \\ &\leq C_2 (1 + 2 \log \omega) \text{Area}(\mathcal{T}_2) \leq C_3 \log \omega \text{Area}(\mathcal{T}_2), \end{aligned} \quad (\text{II.1.29})$$

$$\int_1^\omega L(t) t^{-2} dt \leq \text{Area}(\mathcal{T}_{1,\omega}) \omega^{-2} + 2 \int_1^\omega \text{Area}(\mathcal{T}_{1,t}) t^{-3} dt \leq C_3 \log \omega \text{Area}(\mathcal{T}_2). \quad (\text{II.1.30})$$

Define a (radial) cut-off function  $\eta$  by

$$\eta = \begin{cases} \log \text{dist}_{\Gamma_0}(\sigma_1, \cdot) / \log \omega & \text{on } \mathcal{T}_{1,\omega}, \\ 1 & \text{on } \mathcal{T}_{\omega, R_0/\omega}, \\ [\log R_0 - \log \text{dist}_{\Gamma_0}(\sigma_1, \cdot)] / \log \omega & \text{on } \mathcal{T}_{R_0/\omega, R_0}. \end{cases} \quad (\text{II.1.31})$$

Using the bounds (II.1.29) and (II.1.30), we get

$$\begin{aligned} \int |\nabla \eta|^2 &= \int_{\mathcal{T}_{1,\omega}} |\nabla \eta|^2 + \int_{\mathcal{T}_{R_0/\omega, R_0}} |\nabla \eta|^2 \\ &\leq \frac{1}{(\log \omega)^2} \int_1^\omega \frac{L(t)}{t^2} dt + \frac{1}{(\log \omega)^2} \int_{R_0/\omega}^{R_0} \frac{L(t)}{t^2} dt \leq \frac{C_3 \text{Area}(\mathcal{T}_2)}{\log \omega}. \end{aligned} \quad (\text{II.1.32})$$

Substituting  $\eta \chi$  into the stability inequality, we get using (II.1.25) and (II.1.32)

$$\int_{\mathcal{T}_{\omega, R_0/\omega}} |A|^2 \leq \int_{\mathcal{T}_{R_0} \cap \{\chi < 1\}} |A|^2 + 2 \int_{\mathcal{T}_{R_0}} |\nabla \chi|^2 + 2 \int |\nabla \eta|^2 \leq C_1 R_0 + \frac{2 C_3 \text{Area}(\mathcal{T}_2)}{\log \omega}. \quad (\text{II.1.33})$$

Finally, Lemma II.1.15 (and (II.1.28) for  $t = \omega$ ) gives the first inequality in (II.1.21).  $\square$

We will prove Theorem II.1.2 by considering two separate cases depending on the area of  $\mathcal{T}_1(\sigma)$ . When  $\text{Area}(\mathcal{T}_1(\sigma))$  is small, the next corollary will show that (1) of Theorem II.1.2 holds. On the other hand, when  $\text{Area}(\mathcal{T}_1(\sigma))$  is large, we will show in the next section, using Corollary II.1.45 below, that (2) of Theorem II.1.2 holds.

**Corollary II.1.34.** Given  $C_a$ , there exists  $\Omega_a > 4$  so: Let  $\Gamma \subset \mathbf{R}^3$  be a stable embedded minimal planar domain,  $\sigma = B_1 \cap \partial\Gamma$  connected, and  $\text{dist}_\Gamma(\sigma, \partial\Gamma \setminus \sigma) > R$ . If  $R > \Omega_a^2$  and  $\text{Area}(\mathcal{T}_1(\sigma)) \leq C_a$ , then  $\Gamma$  contains a graph  $\Gamma_g$  (after a rotation) over  $D_{R/\Omega_a} \setminus D_{\Omega_a}$  with gradient  $\leq 1$  and  $\text{dist}_\Gamma(\sigma, \Gamma_g) \leq 2\Omega_a$ .

*Proof.* Lemma II.1.13 gives a simple closed noncontractible curve  $\check{\sigma} \subset \mathcal{T}_{1/32, 1/4}(\sigma)$  and with  $\text{Length}(\check{\sigma}) + \int_{\check{\sigma}} |k_g| \leq C_1 [\text{Area}(\mathcal{T}_1(\sigma)) + 1]$ . Since  $\Gamma$  is a planar domain,  $\check{\sigma}$  separates in  $\Gamma$ ; let  $\check{\Gamma}$  be the component of  $\Gamma \setminus \check{\sigma}$  which does not contain  $\sigma$ . By Lemma II.1.3 (which applies with  $\chi \equiv 1$  since  $\gamma_1 = \gamma_2 = \emptyset$ ), we get for  $1 \leq t \leq R$

$$\text{Area}(\mathcal{T}_t(\check{\sigma}, \check{\Gamma})) \leq C(C_a + 1)t^2. \quad (\text{II.1.35})$$

Given  $\Omega > 4$ , by (II.1.35) and the logarithmic cut-off trick in the stability inequality (cf. (II.1.33)), we get that  $\int_{\mathcal{T}_{\Omega/2, 2R/\Omega}(\check{\sigma}, \check{\Gamma})} |A|^2 \leq C_2(C_a + 1)/\log \Omega$ . Combining this with (II.1.35) and the Cauchy-Schwarz inequality gives for  $\Omega/2 \leq t \leq R/\Omega$

$$\int_{\mathcal{T}_{t, 2t}(\check{\sigma}, \check{\Gamma})} |A| \leq \left( \text{Area}(\mathcal{T}_{2t}(\check{\sigma}, \check{\Gamma})) \int_{\mathcal{T}_{\Omega/2, 2R/\Omega}(\check{\sigma}, \check{\Gamma})} |A|^2 \right)^{1/2} \leq \frac{C_3(C_a + 1)t}{(\log \Omega)^{1/2}}. \quad (\text{II.1.36})$$

Applying the coarea formula on  $\mathcal{T}_{t, 2t}$  for  $t = \Omega/2, R/\Omega$ , (II.1.36) gives a (possibly disconnected) planar domain  $\Gamma_0 \subset \mathcal{T}_{\Omega/2, 2R/\Omega}(\check{\sigma}, \check{\Gamma})$  with  $\mathcal{T}_{\Omega, R/\Omega}(\check{\sigma}, \check{\Gamma}) \subset \Gamma_0$ ,  $\partial\Gamma_0 = \cup_{i=1}^n \sigma_i$ , and

$$\sum_{i=1}^n \int_{\sigma_i} |A| \leq \frac{C_3(C_a + 1)}{(\log \Omega)^{1/2}}. \quad (\text{II.1.37})$$

We now fix  $\Omega = \Omega(C_a) > 4$  so that  $C_2(C_a + 1)/\log \Omega < \pi$  and  $C_3(C_a + 1)(\log \Omega)^{-1/2} < 1/4$ . Using  $\int_{\Gamma_0} |A|^2 < \pi$ , (II.1.37) (which implies equation (1.13) in [CM7] with  $\epsilon = 1/4$ ), and that the Gauss map is conformal, proposition 1.12 of [CM7] implies that on each component  $\Gamma_0^k$  of  $\Gamma_0$  we get  $\mathbf{n}_\Gamma(\Gamma_0^k) \subset \mathcal{B}_{1/2}(a_k)$ , where  $a_k \in \mathbf{S}^2$  ( $\mathbf{n}_\Gamma$  is the unit normal to  $\Gamma$ ). Hence,  $\Gamma_0$  is a (possibly multi-valued) graph. Since  $\Gamma$  is embedded, the corollary now follows easily (cf. lemma 1.10 in [CM4]).  $\square$

We construct next from curves  $\sigma_1, \gamma_1, \gamma_2$  in a stable surface the desired multi-valued graph. (The existence of the curves  $\sigma_1, \gamma_1, \gamma_2$  will be established in the next section.) First we need the following two lemmas:

**Lemma II.1.38.** Given  $C_1, \epsilon_0 > 0$ , there exists  $\epsilon_1 > 0$  so if  $\mathcal{B}_1 \subset \Sigma$  is minimal,  $\sup_{\mathcal{B}_{1/2}} |A|^2 \leq \epsilon_1$ , and  $\sup_{\mathcal{B}_1} |A|^2 \leq C_1$ , then  $\sup_{\mathcal{B}_{3/4}} |A|^2 \leq \epsilon_0$ .

*Proof.* Suppose not; it follows that there is a sequence  $\Sigma_j$  of minimal surfaces with  $\sup_{\mathcal{B}_{1/2}} |A|^2 \leq 1/j$ ,  $\sup_{\mathcal{B}_1} |A|^2 \leq C_1$ , and  $\sup_{\mathcal{B}_{3/4}} |A|^2 > \epsilon_0 > 0$ . The uniform bound  $\sup_{\mathcal{B}_1} |A|^2 \leq C_1$  (and standard elliptic estimates) gives a subsequence which converges in  $C^{2, \alpha}$  to a limit  $\Sigma_\infty$ . It follows that  $\Sigma_\infty$  is minimal,  $|A|^2 = 0$  on  $\mathcal{B}_{1/2}$ , and  $\sup_{\mathcal{B}_{3/4}} |A|^2 \geq \epsilon_0 > 0$ . By unique continuation,  $\Sigma_\infty$  is flat contradicting that  $\sup_{\mathcal{B}_{3/4}} |A|^2 \geq \epsilon_0 > 0$ .  $\square$

The next lemma will be applied both when  $\Gamma$  is an annulus and when  $\Gamma$  has boundary on the sides. When  $\Gamma$  is an annulus, the condition (II.1.40) will be trivially satisfied and it will be possible for  $\Gamma$  to contain a graph instead of a multi-valued graph.

**Lemma II.1.39.** Given  $N, S_0 > 4$ ,  $\epsilon > 0$ , there exist  $C_b > 1$ ,  $\delta > 0$  so: Let  $\Gamma \subset \mathbf{R}^3$  be a stable embedded minimal surface and  $\sigma = B_1 \cap \partial\Gamma$ . If  $\gamma : [0, S_0] \rightarrow \Gamma$  is a geodesic with  $\text{dist}_\Gamma(\gamma(t), \sigma) = t$  for  $0 \leq t \leq S_0$ ,  $\sup_{\mathcal{B}_{S_0/16}(\gamma(S_0))} |A|^2 \leq \delta S_0^{-2}$ , and for  $0 \leq t \leq S_0$

$$\text{dist}_{\Gamma \setminus \mathcal{I}_{t/8}(\sigma)}(\gamma(t), \partial\Gamma) \geq C_b t, \quad (\text{II.1.40})$$

then (after a rotation of  $\mathbf{R}^3$ )  $\Gamma$  contains either an  $N$ -valued graph  $\Gamma_{2, S_0/2}^{-N\pi, N\pi}$ , or a graph  $\Gamma_{2, S_0/2}$ , with gradient  $\leq \epsilon$ ,  $|A| \leq \epsilon/r$ , and  $\gamma(4) \in \Gamma_{2, 5}^{-\pi, \pi}$  (or in  $\Gamma_{2, 5}$ ).

*Proof.* Combining estimates for stable surfaces of [Sc], [CM2] and (II.1.40), gives for  $0 \leq t \leq S_0$

$$\sup_{\mathcal{B}_{t/2}(\gamma(t))} |A| \leq C_0 t^{-1}. \quad (\text{II.1.41})$$

Fix  $\delta_0 > 0$  to be chosen small depending on  $S_0$ . Using (II.1.41) and repeatedly applying Lemma II.1.38 along a chain of balls with centers in  $\gamma$ , there exists  $\delta_1 = \delta_1(S_0, \delta_0, C_0) > 0$  so that if  $\delta \leq \delta_1$ , then for  $1 \leq t \leq S_0$

$$\sup_{\mathcal{B}_{t/32}(\gamma(t))} |A| \leq \delta_0 t^{-1}. \quad (\text{II.1.42})$$

Since  $\gamma$  is a geodesic in  $\Gamma$ , (II.1.42) gives the bound  $k_g^{\mathbf{R}^3}(t) \leq \delta_0 t^{-1}$  for the geodesic curvature of  $\gamma$  in  $\mathbf{R}^3$ . It follows that for  $1 \leq t \leq S_0$

$$|\mathbf{n}_\Gamma(\gamma(t)) - \mathbf{n}_\Gamma(\gamma(1))| + |\gamma'(t) - \gamma'(1)| \leq 2\delta_0 \int_1^{S_0} \frac{ds}{s} \leq 2\delta_0 \log S_0; \quad (\text{II.1.43})$$

i.e.,  $\gamma$  is  $C^1$ -close to a straight line segment in  $\mathbf{R}^3$  and  $\mathbf{n}_\Gamma$  is almost constant on  $\gamma$ . Rotate so that  $\gamma'(1) = (1, 0, 0)$  (i.e., so  $\gamma'(1)$  points in the  $x_1$ -direction). For  $\delta_0 > 0$  small, (II.1.43) (and  $\gamma(0) \in B_1$ ) implies that for  $1 \leq t \leq S_0$

$$3t/4 - 2 \leq x_1(\gamma(t)) \leq 1 + t. \quad (\text{II.1.44})$$

We will now argue as in (II.1.41)–(II.1.42) to extend the region where  $\Gamma$  is graphical, this time using balls centered on cylinders (i.e., building out the multi-valued graph in the  $\theta$  direction). Suppose now that  $4 \leq s \leq S_0/2$  and  $y_{0,s} = \{x_1^2 + x_2^2 = s^2\} \cap \gamma$ . Using (II.1.42),  $\mathcal{B}_{C_2 s}(y_{0,s})$  is a graph with gradient  $\leq C_2' \delta_0$  over  $\mathbf{n}_\Gamma(y_{0,s})$ . In particular, also using (II.1.43),  $\partial\mathcal{B}_{C_2 s}(y_{0,s})$  contains a point  $y_{1,s} \in \{x_1^2 + x_2^2 = s^2\}$ . Using Lemma II.1.38, we can therefore repeat this to find  $y_{2,s}$ , etc. It follows from (II.1.40) that we can continue this until  $\Gamma$  either closes up (giving a graph) or we have the desired  $N$ -valued graph  $\Gamma_{2, S_0/2}^{-N\pi, N\pi}$  with gradient  $\leq \epsilon$ ,  $|A| \leq \epsilon/r$ , and which contains  $\gamma(4)$ .  $\square$

In the next corollary  $\Gamma \subset B_{2R} \subset \mathbf{R}^3$  will be a stable embedded minimal annulus with  $\partial\Gamma \subset B_{1/4} \cup \partial B_{2R}$  where  $B_1 \cap \partial\Gamma$  is connected and suppose  $\Gamma_0 \subset \Gamma$  is a disk satisfying (A) for  $C_0 = 0$ , (B), (D). Let  $\sigma = B_1 \cap \partial\Gamma$  so  $\sigma_1 \subset \sigma$  and  $\sigma$  is a simple closed curve. Assume also that the following strengthening of (A) holds:

(A')  $\text{dist}_\Gamma(\gamma_i(t), \sigma) = t$  for  $0 \leq t \leq \text{Length}(\gamma_i)$ .

**Corollary II.1.45.** Given  $N, \epsilon > 0$ , there exist  $\omega_0, R_0 > 1$  so if  $\Gamma, \Gamma_0$  are as above, and  $\text{Area}(\mathcal{T}_1(\sigma_1)) \geq 4R_0^2(R_0 + 1)$ , then (after a rotation of  $\mathbf{R}^3$ )  $\Gamma$  contains an  $N$ -valued graph  $\Gamma_{\omega_0, R/\omega_0}^{-N\pi, N\pi}$  with gradient  $\leq \epsilon$ ,  $|A| \leq \epsilon/r$ , and

$$\text{dist}_\Gamma(z_1, \Gamma_{\omega_0, \omega_0}^{0,0}) < 2\omega_0 + C_1 \text{Area}(\mathcal{T}_1(\sigma_1, \Gamma)). \quad (\text{II.1.46})$$

*Proof.* Proposition II.1.20 gives  $C$  so that for  $\omega \leq t \leq R_0/\omega$  (where  $\omega > 4$  and  $R_0^2 > 2\omega^2$ )

$$\text{Area}(\mathcal{T}_2(\sigma_1, \Gamma_0)) t^2 / C \leq \text{Area}(\mathcal{T}_{\omega, t}(\sigma_1, \Gamma_0)), \quad (\text{II.1.47})$$

$$\int_{\mathcal{T}_{\omega, R_0/\omega}(\sigma_1, \Gamma_0)} |A|^2 \leq \frac{C}{\log \omega} \text{Area}(\mathcal{T}_2(\sigma_1, \Gamma_0)). \quad (\text{II.1.48})$$

(Here we also used  $\text{Area}(\mathcal{T}_1(\sigma_1)) \geq 4R_0^2(R_0 + 1)$  in (II.1.48).) Set  $S = \omega$ . Choose a maximal disjoint collection of balls  $\mathcal{B}_{S/4}(y_1), \dots, \mathcal{B}_{S/4}(y_n)$  with centers in  $\mathcal{T}_{S, 2S}(\sigma_1, \Gamma_0)$ . Since  $\Gamma$  is annulus without boundary on the sides and  $R_0 > 5S/2$ , it follows from (A') that  $\mathcal{B}_{S/2}(y_j) \cap \partial\Gamma = \emptyset$ ; we use this twice. First, since  $\mathcal{T}_{S, 2S}(\sigma_1, \Gamma_0)$  is contained in the union of the double balls and  $\pi(S/4)^2 \leq \text{Area}(\mathcal{B}_{S/4}(y_j)) \leq \text{Area}(\mathcal{B}_{S/2}(y_j)) \leq C\pi S^2$  by stability (see [CM2]), we have  $n \geq C S^{-2} \text{Area}(\mathcal{T}_{S, 2S}(\sigma_1, \Gamma_0))$ . Second, again by stability, [CM2],  $\int_{\mathcal{T}_{S/4}(\gamma_1 \cup \gamma_2) \cap \mathcal{T}_{S/2, 3S}(\sigma_1)} |A|^2 \leq C$ . Combining this with (II.1.47) and (II.1.48), we can find  $j$  so that  $\int_{\mathcal{B}_{S/4}(y_j)} |A|^2 < C/\log \omega$ , and therefore, by the mean value inequality,  $\sup_{\mathcal{B}_{S/8}(y_j)} |A|^2 < C S^{-2}/\log \omega$ .

Let  $\gamma : [0, \ell] \rightarrow \Gamma$  be a minimal geodesic from  $y_j$  to  $\sigma_1$ ; note that  $S \leq \ell \leq 2S$ . Using that the sides  $\gamma_1, \gamma_2$  are minimizing (i.e., (A')), it follows that  $\gamma \subset \Gamma_0$ . Furthermore, since  $\Gamma$  is an annulus, (A') implies that

$$\text{dist}_{\Gamma \setminus \mathcal{T}_1(\sigma)}(\gamma(\ell), \partial\Gamma) \geq R/2, \quad (\text{II.1.49})$$

In particular, given  $\omega_1, N_1 > 1$  and  $\epsilon_1 > 0$ , there exists  $\omega$  (and hence  $R_0$ ) large so we can apply Lemma II.1.39 to get either a graph  $\Gamma_{S/\omega_1, S/2}$  or an initial multi-valued graph  $\Gamma_{S/\omega_1, S/2}^{-N_1\pi, N_1\pi}$  with gradient  $\leq \epsilon_1$ ,  $|A| \leq \epsilon_1/r$ , and  $\gamma(4S/\omega_1) \in \Gamma_{2S/\omega_1, 5S/\omega_1}^{-\pi, \pi}$ . However, since  $\text{Area}(\mathcal{T}_1(\sigma_1)) \geq 4R_0^2(R_0 + 1)$ ,  $\Gamma$  cannot contain a graph  $\Gamma_{S/\omega_1, S/2}$ .

Using theorem II.0.21 of [CM3], we will next extend  $\Gamma_{S/\omega_1, S/2}^{-N_1\pi, N_1\pi}$  to the desired  $N$ -valued graph  $\Gamma_{\omega_0, R/\omega_0}^{-N\pi, N\pi}$ . Namely, let  $P$  be the vertical plane  $\{x_1 = 2S/\omega_1\}$ . We claim first that each component of  $P \cap \Gamma$  goes off to  $\partial B_{2R}$ . To see this, note that by the maximum principle, any closed curve in  $P \cap \Gamma$  would be homologous to the interior boundary of  $\Gamma$  and together these two curves would span an annulus in  $\Gamma$  violating the convex hull property (using the multi-valued graph in  $\Gamma$  to connect this annulus to  $\{x_1 = -S/\omega_1\}$ ). It follows that two of these nodal curves connect the multi-valued graph out to  $\partial B_{2R}$ , giving a curve  $\eta$  in  $\Gamma$  with both endpoints in  $\partial B_{2R}$ . One component of  $\Gamma \setminus \eta$  is a stable disk which is forced to spiral initially. Therefore, by theorem II.0.21 of [CM3], this extends to the desired multi-valued graph.  $\square$

## II.2. THE MINIMIZING GEODESICS AND THE PROOF OF THEOREM II.1.2

In Proposition II.2.9 and Corollary II.2.10 below, we will construct the minimizing geodesics  $\gamma_1, \gamma_2$  needed for Corollary II.1.45. To do that we will first need the following lemmas and corollaries (here  $\mathcal{T}_t$  is the closed tubular neighborhood and  $\mathcal{T}_t^\circ$  is the open):

**Lemma II.2.1.** Let  $\Sigma$  be an annulus with  $\partial\Sigma = \sigma_1 \cup \sigma_2$ , where  $\text{dist}_\Sigma(\sigma_1, \sigma_2) > \ell + \epsilon$  for  $\ell, \epsilon > 0$  and let  $E$  be the connected component of  $\Sigma \setminus \mathcal{T}_\ell(\sigma_1)$  containing  $\sigma_2$ . Let  $\gamma_1, \gamma_2$  be geodesics with

$$\gamma_i : [0, \ell] \rightarrow \Sigma, \text{dist}_\Sigma(\gamma_i(t), \sigma_1) = t \text{ for } 0 \leq t \leq \ell, \text{ and } \gamma_i(\ell) \in \overline{E}. \quad (\text{II.2.2})$$

If  $\sigma_3 \subset \sigma_1$  is a segment connecting  $\gamma_1(0)$  and  $\gamma_2(0)$ , then there exists a curve  $\sigma_4 \subset \mathcal{T}_\epsilon^\circ(E) \cap \mathcal{T}_\epsilon^\circ(\Sigma \setminus E)$  connecting  $\gamma_1(\ell)$  and  $\gamma_2(\ell)$  and so  $\sigma_3 \cup \sigma_4 \cup \gamma_1 \cup \gamma_2$  bounds a disk  $\Sigma_4$ . Moreover,  $\sigma_4 \subset \mathcal{T}_{\ell+\epsilon}(\sigma_1) \setminus \mathcal{T}_{\ell-\epsilon}(\sigma_1)$ .

*Proof.* First, note that  $\gamma_1(\ell), \gamma_2(\ell) \in \overline{E} \cap \overline{\Sigma \setminus E}$  and by definition  $E$ , hence  $\mathcal{T}_\epsilon^\circ(E)$ , is connected. Moreover, if  $x \in \Sigma \setminus E$  and  $\gamma : [0, \ell_\gamma] \rightarrow \Sigma$  is a geodesic with  $\gamma(\ell_\gamma) = x$  and  $\text{dist}_\Sigma(\gamma(t), \sigma_1) = t$  for  $0 \leq t \leq \ell_\gamma$ , then  $\gamma \cap E = \emptyset$ . Hence, also  $\Sigma \setminus E$  and  $\mathcal{T}_\epsilon^\circ(\Sigma \setminus E)$  are connected. Since  $\sigma_1 \subset \mathcal{T}_\epsilon^\circ(\Sigma \setminus E)$  and  $\sigma_2 \subset \mathcal{T}_\epsilon^\circ(E)$ , applying van Kampen's theorem to  $\Sigma = \mathcal{T}_\epsilon^\circ(\Sigma \setminus E) \cup \mathcal{T}_\epsilon^\circ(E)$  gives that  $\mathcal{T}_\epsilon^\circ(E) \cap \mathcal{T}_\epsilon^\circ(\Sigma \setminus E)$  is path connected and has fundamental group  $\mathbf{Z}$  which injects into  $\pi_1(\Sigma)$ . In particular, we get simple curves  $\sigma_{4,1}, \sigma_{4,2} \subset \mathcal{T}_\epsilon^\circ(E) \cap \mathcal{T}_\epsilon^\circ(\Sigma \setminus E)$  connecting  $\gamma_1(\ell)$  to  $\gamma_2(\ell)$  and so  $\sigma_{4,1} \cup \sigma_{4,2}$  is homologous to  $\sigma_1$ . Fix  $\Sigma_0 \subset \Sigma$  with  $\partial\Sigma_0 = \sigma_1 \cup (\sigma_{4,1} \cup \sigma_{4,2})$ . The curve  $\sigma_3 \cup \gamma_1 \cup \gamma_2$  divides  $\Sigma_0$  into two components, one of which is a disk with  $\sigma_3, \gamma_1, \gamma_2$ , and either  $\sigma_{4,1}$  or  $\sigma_{4,2}$  in its boundary.

Finally, since  $\sigma_4 \subset \mathcal{T}_\epsilon^\circ(E)$  it follows that  $\sigma_4 \subset \Sigma \setminus \mathcal{T}_{\ell-\epsilon}(\sigma_1)$ . Likewise it follows from that  $\sigma_4 \subset \mathcal{T}_\epsilon^\circ(E) \cap \mathcal{T}_\epsilon^\circ(\Sigma \setminus E)$  and the triangle inequality that  $\sigma_4 \subset \mathcal{T}_{\ell+\epsilon}(\sigma_1)$ .  $\square$

**Corollary II.2.3.** Let  $\Sigma, E, \sigma_1, \sigma_2, \sigma_3, \gamma_1, \gamma_2$  be as in Lemma II.2.1. If  $\gamma_1(\ell) \neq \gamma_2(\ell)$ , then there exists a geodesic  $\gamma_3$  different from  $\gamma_1, \gamma_2$ , intersecting  $\sigma_3$ , and satisfying (II.2.2).

*Proof.* Let  $\eta \subset \Sigma \setminus (\gamma_1 \cup \gamma_2)$  be a simple curve from  $\sigma_3$  to  $\sigma_2$  and so  $\eta \cap \sigma_1 \subset \eta \cap \sigma_3$  is one point. Fix  $\mu > 0$  with  $3\mu < \text{dist}_\Sigma(\gamma_1 \cup \gamma_2, \eta)$ . For  $\epsilon > 0$  small (in particular,  $\epsilon < \text{dist}_\Sigma(\sigma_1, \sigma_2) - \ell$ ), let  $\sigma_{\epsilon,4}, \Sigma_{\epsilon,4}$  be given by Lemma II.2.1. Let  $\eta_\epsilon$  be the component of  $\eta \cap \Sigma_{\epsilon,4}$  intersecting  $\sigma_3$  and let  $\gamma_{\epsilon,3} : [0, \ell_\epsilon] \rightarrow \Sigma$  be a geodesic with  $\gamma_{\epsilon,3}(\ell_\epsilon) = \partial\eta_\epsilon \setminus \sigma_1$  and  $\text{dist}_\Sigma(\gamma_{\epsilon,3}(t), \sigma_1) = t$  for  $0 \leq t \leq \ell_\epsilon$ . Since  $\sigma_{\epsilon,4} \subset \mathcal{T}_{\ell+\epsilon}(\sigma_1) \setminus \mathcal{T}_{\ell-\epsilon}(\sigma_1)$ , then  $\ell - \epsilon < \ell_\epsilon \leq \ell + \epsilon$ . Moreover, by the triangle inequality, if  $\epsilon < \mu$ , then  $\mathcal{B}_\mu(\gamma_k(\ell)) \cap \gamma_{\epsilon,3} = \emptyset$  for  $k = 1, 2$ , hence  $\mathcal{B}_\mu(\gamma_k(\ell)) \cap (\eta \cup \gamma_{\epsilon,3}) = \emptyset$  for  $k = 1, 2$  and  $\epsilon < \mu$ . We claim that

$$\eta_\epsilon \cup \gamma_{\epsilon,3} \subset \mathcal{T}_\delta(\Sigma \setminus E) \text{ where } \delta \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \quad (\text{II.2.4})$$

Suppose that (II.2.4) fails; it follows that there exists a sequence  $\epsilon_i \rightarrow 0$  and  $x_i \in \eta_{\epsilon_i}$  with  $x_i \rightarrow x$ , where  $x_i, x \in \Sigma \setminus \mathcal{T}_\delta^\circ(\Sigma \setminus E) \subset E$  for some  $\delta > 0$ . Since  $E$  is open and connected, there exists a curve  $\nu \subset E$  from  $x$  to  $\sigma_2$  and so  $\nu \subset E \setminus \mathcal{T}_{\delta_0}(\Sigma \setminus E)$  for some  $\delta_0 = \delta_0(x) > 0$ . For  $i$  sufficiently large, we can extend  $\nu$  to a curve  $\nu_i \subset E \setminus \mathcal{T}_{\delta_0/2}(\Sigma \setminus E)$  from  $x_i$  to  $\sigma_2$ . However,  $\gamma_1 \cup \gamma_2 \cup \sigma_{\epsilon_i,4} \subset \mathcal{T}_{\epsilon_i}(\Sigma \setminus E)$  separates  $x_i$  from  $\sigma_2$  which is a contradiction for  $i$  sufficiently large. Hence, (II.2.4) holds.

Pick a sequence  $\epsilon_i > 0$  with  $\epsilon_i \rightarrow 0$ . After passing to a subsequence, we can assume that  $\gamma_{\epsilon_j,3} \rightarrow \gamma_3$ . It is clear that  $\gamma_3 : [0, \ell] \rightarrow \Sigma$  is a geodesic with  $\gamma_3(0) \in \sigma_1 \setminus \{\gamma_1(0), \gamma_2(0)\}$ ,  $\text{dist}_\Sigma(\gamma_3(t), \sigma_1) = t$  for  $0 \leq t \leq \ell$ , and  $\gamma_3(\ell) \in \overline{E}$ . It remains to show that  $\gamma_3(0) \in \sigma_3$ .

If  $\gamma_3(0) \notin \sigma_3$ , then  $\text{dist}_\Sigma(\gamma_3(0), \sigma_3) > 0$  (since  $\gamma_3(0) \in \sigma_1 \setminus \{\gamma_1(0), \gamma_2(0)\}$ ) and therefore  $\text{dist}_\Sigma(\gamma_{\epsilon_j,3}(0), \sigma_3) > 0$  for  $j$  large. It follows that  $\eta_{\epsilon_j} \cup \gamma_{\epsilon_j,3}$  divides  $\Sigma$  into two components  $\Sigma_{\epsilon_j,1}, \Sigma_{\epsilon_j,2}$  with  $\gamma_1(\ell) \in \Sigma_{\epsilon_j,1}$  and  $\gamma_2(\ell) \in \Sigma_{\epsilon_j,2}$ . (That  $\gamma_1(\ell), \gamma_2(\ell)$  are in different components follows from that  $\gamma_{\epsilon,3} \cap \gamma_1 = \emptyset = \gamma_{\epsilon,3} \cap \gamma_2$  by the triangle inequality.) After possibly switching  $\gamma_1$  and  $\gamma_2$  (and going to a subsequence), we can assume that  $\sigma_2 \subset \Sigma_{\epsilon_j,2}$ . Note that,

$\mathcal{B}_\mu(\gamma_1(\ell)) \subset \Sigma_{\epsilon_j,1}$  since we showed above that  $\mathcal{B}_\mu(\gamma_1(\ell)) \cap (\eta \cup \gamma_{\epsilon,3}) = \emptyset$ . We will use this to contradict that  $\gamma_1(\ell) \in \bar{E}$ . Namely, let  $x \in \mathcal{B}_{\mu/2}(\gamma_1(\ell)) \cap E$  (note that such an  $x$  exists since  $\gamma_1(\ell) \in \bar{E}$ ). Since  $E$  is open and connected, then there exists a curve  $\nu \subset E \setminus \mathcal{T}_{\delta_0}(\Sigma \setminus E)$  for some sufficiently small  $\delta_0 = \delta_0(x) > 0$  which connect  $x$  and  $\sigma_2$ . This contradicts (II.2.4) for  $j$  sufficiently large since  $\eta_{\epsilon_j} \cup \gamma_{\epsilon_j,3}$  separate  $\Sigma_{\epsilon_j,1}$  and  $\sigma_2$ .  $\square$

**Lemma II.2.5.** If  $\Sigma \subset \mathbf{R}^3$  is an immersed minimal surface with  $\partial\Sigma = \gamma_1 \cup \gamma_2 \cup \sigma$  where  $\sigma \subset B_1$ ,  $\partial_{\mathbf{n}}|x| \geq 0$  on  $\sigma$  ( $\mathbf{n}$  is the inward normal to  $\partial\Sigma$ ), and  $\gamma_1, \gamma_2$  have length  $\leq \ell$ , then

$$\text{Area}(\mathcal{T}_1(\sigma)) \leq 4\ell(\ell + 1). \quad (\text{II.2.6})$$

*Proof.* By minimality, Stokes' theorem,  $\partial_{\mathbf{n}}|x| \geq 0$  on  $\sigma$ , and  $|x| \leq \ell + 1$  on  $\gamma_i$ ,

$$4 \text{Area}(\Sigma) = \int_{\Sigma} \Delta|x|^2 \leq 2 \int_{\gamma_1 \cup \gamma_2} |x| |\partial_{\mathbf{n}}|x|| \leq 4(\ell + 1)\ell. \quad (\text{II.2.7})$$

$\square$

In what follows, if  $\sigma \subset \partial\Sigma$  is a simple curve,  $\mathbf{n}$  is the inward normal to  $\sigma$ ,  $\tilde{\sigma}$  is a segment of  $\sigma$ , then

$$\mathcal{T}_s(\tilde{\sigma}, \mathbf{n}) = \{\exp_{\tilde{\sigma}(t)}(\tau \mathbf{n}(t)) \mid \text{dist}_{\Sigma}(\exp_{\tilde{\sigma}(t)}(\tau \mathbf{n}(t)), \sigma) = \tau \leq s\}. \quad (\text{II.2.8})$$

**Proposition II.2.9.** Let  $\Sigma \subset \mathbf{R}^3$  be an immersed minimal annulus,  $\sigma \subset B_1 \cap \partial\Sigma$  a simple closed curve with  $\text{dist}_{\Sigma}(\sigma, \partial\Sigma \setminus \sigma) > \ell$ ,  $\partial_{\mathbf{n}}|x| \geq 0$  on  $\sigma$ , and let  $E$  be as in Lemma II.2.1. If  $\tilde{\sigma}$  is a segment of  $\sigma$  and  $\text{Area}(\mathcal{T}_1(\tilde{\sigma}, \mathbf{n})) > 4\ell(\ell + 1)$ , then there exists a geodesic  $\gamma : [0, \ell] \rightarrow \Sigma$  with  $\text{dist}_{\Sigma}(\gamma(t), \sigma) = t$  for  $0 \leq t \leq \ell$  and  $\gamma(0) \in \tilde{\sigma}$ ,  $\gamma(\ell) \in \bar{E}$ .

*Proof.* Suppose that there is no such geodesic  $\gamma$ . Let  $B$  be the set of geodesics satisfying (II.2.2) for  $\sigma_1 = \sigma$ . It follows easily that  $A = \{\gamma_0(0) \mid \gamma_0 \in B\}$  is a closed subset of  $\sigma \setminus \tilde{\sigma}$  containing more than two points. Let  $\hat{\sigma}$  be the connected component of  $\sigma \setminus A$  containing  $\tilde{\sigma}$  (note that  $\hat{\sigma}$  is open) and let  $\partial\hat{\sigma} = \{\gamma_1(0), \gamma_2(0)\}$  where  $\gamma_1, \gamma_2$  are the corresponding minimizing geodesics of lengths  $\ell$ .

By Corollary II.2.3,  $\gamma_1(\ell) = \gamma_2(\ell)$ . In fact, there exists a subset  $\hat{\Sigma}$  of  $\Sigma$  with  $\partial\hat{\Sigma} = \gamma_1 \cup \gamma_2 \cup \hat{\sigma}$ . Since  $\text{Area}(\mathcal{T}_1(\hat{\sigma}, \hat{\Sigma})) \geq \text{Area}(\mathcal{T}_1(\tilde{\sigma}, \tilde{\Sigma})) = \text{Area}(\mathcal{T}_1(\tilde{\sigma}, \mathbf{n})) > 4\ell(\ell + 1)$ , it follows from Lemma II.2.5 that  $A \cap \tilde{\sigma} \neq \emptyset$  which is the desired contradiction and the proposition follows.  $\square$

**Corollary II.2.10.** Let  $\Sigma \subset \mathbf{R}^3$  be an immersed minimal annulus,  $\sigma \subset B_1 \cap \partial\Sigma$  is a simple closed curve with  $\text{dist}_{\Sigma}(\sigma, \partial\Sigma \setminus \sigma) > \ell \geq 1$ ,  $\partial_{\mathbf{n}}|x| \geq 0$  on  $\sigma$ , and  $\text{Area}(\mathcal{T}_1(\sigma, \Sigma)) > 12\ell^2(\ell + 1)$ . For each  $z_1 \in \sigma$  there is a segment  $z_1 \in \sigma_1 \subset \sigma$  and geodesics  $\gamma_1, \gamma_2 : [0, \ell] \rightarrow \Sigma$  with  $\{\gamma_1(0), \gamma_2(0)\} = \partial\sigma_1$ ,

$$\text{dist}_{\Sigma}(\gamma_i(t), \sigma) = t \text{ for } 0 \leq t \leq \ell. \quad (\text{II.2.11})$$

Moreover, for all  $\epsilon > 0$  a disk  $\Sigma_0 \subset \Sigma$  has  $\sigma_1, \gamma_i \subset \partial\Sigma_0$ ,  $\text{dist}_{\Sigma_0}(\partial\Sigma_0 \setminus \sigma_1 \cup \gamma_1 \cup \gamma_2) > \ell - \epsilon$ ,

$$15\ell^2(\ell + 1) > \text{Area}(\mathcal{T}_1(\sigma_1, \Sigma_0)) > 4\ell^2(\ell + 1). \quad (\text{II.2.12})$$

*Proof.* Let  $\sigma_{z_1}^1, \sigma_{z_1}^2, \sigma_{z_1}^3$  be three consecutive (disjoint) subsegments of  $\sigma$  with  $z_1 \in \sigma_{z_1}^2$  being the ‘‘middle one’’ and so for each  $i$

$$5\ell^2(\ell + 1) > \text{Area}(\mathcal{T}_1(\sigma_{z_1}^i, \mathbf{n})) > 4\ell^2(\ell + 1). \quad (\text{II.2.13})$$

By Proposition II.2.9 applied to both  $\sigma_{z_1}^1$  and  $\sigma_{z_1}^3$ , we get geodesics  $\gamma_1, \gamma_2 : [0, \ell] \rightarrow \Sigma$  satisfying (II.2.11) and with  $\gamma_1(0) \in \sigma_{z_1}^1, \gamma_2(0) \in \sigma_{z_1}^3, \gamma_i(\ell) \in \overline{E}$  (where  $E$  is the connected component of  $\Sigma \setminus \mathcal{T}_\ell(\sigma)$  containing  $\sigma_2$ ). Let  $\sigma_1$  be the segment of  $\sigma$  between  $\gamma_1(0)$  and  $\gamma_2(0)$  and containing  $\sigma_{z_1}^2$ . By Lemma II.2.1 there is a disk  $\Sigma_0 \subset \Sigma$  with  $\sigma_1, \gamma_1, \gamma_2 \subset \partial\Sigma_0$ , and  $\text{dist}_{\Sigma_0}(\partial\Sigma_0 \setminus \sigma_1 \cup \gamma_1 \cup \gamma_2) > \ell - \epsilon$ . We need to show (II.2.12). Since  $\sigma_{z_1}^2 \subset \sigma_1$  the lower bound in (II.2.12) follows easily from (II.2.13). To see the upper bound, observe that if  $x \in \mathcal{T}_1(\sigma_1, \Sigma_0)$ , then clearly  $\text{dist}_\Sigma(x, \sigma) = \text{dist}_{\Sigma_0}(x, \sigma_1)$  and hence

$$\mathcal{T}_1(\sigma_1, \Sigma_0) \subset \cup_{i=1,2,3} \mathcal{T}_1(\sigma_{z_1}^i, \mathbf{n}). \quad (\text{II.2.14})$$

From (II.2.13) and (II.2.14), the upper bound in (II.2.12) follows.  $\square$

*Proof.* (of Theorem II.1.2). Given  $N, \epsilon$ , let  $\omega_0, R_0$  be given by Corollary II.1.45. Set  $\sigma = \partial B_1 \cap \partial \Gamma_1(\partial)$  and note that  $\partial_{\mathbf{n}}|x| \geq 0$  and  $B_1 \cap \partial \Gamma$  is connected since  $\Gamma$  is an annulus. Suppose that  $\text{dist}_\Sigma(\sigma, \partial \Gamma \setminus \sigma) > R_0$ . By Corollary II.1.34, (1) holds if  $\text{Area}(\mathcal{T}_1(\sigma)) \leq 12 R_0^2 (R_0 + 1)$ . (Recall that if one component of  $B_{R/\omega} \cap \Gamma \setminus B_\omega$  contains a graph over  $D_{R/(2\omega)} \setminus D_{2\omega}$  with gradient  $\leq 1$ , then every component of  $B_{R/(C\omega)} \cap \Gamma \setminus B_{C\omega}$  is a graph for some  $C > 1$ .)

On the other hand if  $\text{Area}(\mathcal{T}_1(\sigma)) > 12 R_0^2 (R_0 + 1)$ , then it follows from Corollary II.1.45 together with Corollary II.2.10 that (2) holds.  $\square$

Using that the curvature of a 2-valued embedded minimal graph decays faster than quadratically (this was shown in [CM8]), we show next (which will be needed in the next section) that such 2-valued graphs contain minimal geodesics close to the radial curve  $\theta = 0$ . (In particular, there is such a geodesic which do not spiral.) In this corollary,  $\Gamma_{\lambda\omega}(\partial)$  denotes the component of  $B_{\lambda\omega} \cap \Gamma$  containing  $B_{\lambda\omega} \cap \partial \Gamma$ .

**Corollary II.2.15.** There exists  $\lambda > 1$  so: If  $\Gamma$  is as in Theorem II.1.2,  $\Gamma_{\omega, R/\omega}^{-3\pi, 3\pi}$  is as in (2) of that theorem (with  $N \geq 3, \tau \leq 1$ ), and  $R > \lambda\omega^2$ , then there exists a geodesic  $\gamma : [0, \ell] \rightarrow \Gamma_{\omega, R^{1/2}}^{-\pi, \pi}$  with  $\gamma(0) \in \partial B_{\lambda\omega}, \text{dist}_\Gamma(\gamma(t), \Gamma_{\lambda\omega}(\partial)) = t$  for  $0 \leq t \leq \ell$ , and  $\ell \geq R^{1/2}/4$ .

*Proof.* Fix  $\lambda > 1$  large to be chosen. Set  $r = \text{dist}_{\Gamma_{\omega, R^{1/2}}^{-2\pi, 2\pi}}(\Gamma_{\omega, \omega}^{-2\pi, 2\pi}, \cdot)$  and let  $\Gamma_{\omega, R/\omega}^{-3\pi, 3\pi}$  be the graph of  $u$ . By Corollary 1.14 of [CM8], on  $S_{\omega, R^{1/2}}^{-2\pi, 2\pi}$  we have  $\rho |\text{Hess}_u| \leq C' (\rho/\omega)^{-5/12}$ . Hence, on  $\Gamma_{\omega, R^{1/2}}^{-2\pi, 2\pi}$

$$r |A| \leq C \omega^{5/12} r^{-5/12}. \quad (\text{II.2.16})$$

Let  $\gamma : [0, \ell] \rightarrow \Gamma$  be a minimizing geodesic in  $\Gamma$  from (the point)  $\Gamma_{R^{1/2}/3, R^{1/2}/3}^{0,0}$  to  $\Gamma_{\lambda\omega}(\partial)$ , so  $\text{dist}_\Gamma(\gamma(t), \Gamma_{\lambda\omega}(\partial)) = t$  for  $0 \leq t \leq \ell$ . In particular,  $\gamma(0) \in \partial \Gamma_{\lambda\omega}(\partial)$  and  $\gamma(\ell) = \Gamma_{R^{1/2}/3, R^{1/2}/3}^{0,0}$ . Using the radial curve  $\Gamma_{\omega, R^{1/2}/3}^{0,0}$  as a comparison (and that  $\tau \leq 1$ ),  $\text{Length}(\gamma) = \ell \leq R^{1/2}/2$ . Let  $\tilde{\gamma}$  be the maximal segment of  $\gamma$  in  $\Gamma_{\omega, R^{1/2}}^{-\pi, \pi}$  containing  $\gamma(\ell)$ . Since  $\tilde{\gamma}$  is a geodesic in  $\Gamma$ , (II.2.16) gives the bound  $k_g^{\mathbf{R}^3}(t) \leq C \omega^{5/12} t^{-1-5/12}$  for the geodesic curvature of  $\tilde{\gamma}$  in  $\mathbf{R}^3$ . It follows that for  $\lambda\omega \leq t \leq \ell$

$$|\tilde{\gamma}'(t) - \gamma'(\ell)| \leq C \omega^{5/12} \int_{\lambda\omega}^{\infty} s^{-17/12} ds \leq 12 C \lambda^{-5/12}/5; \quad (\text{II.2.17})$$

i.e.,  $\tilde{\gamma}$  is  $C^1$ -close to a straight line segment in  $\mathbf{R}^3$ . For  $\lambda$  large, (II.2.17) implies that either  $\tilde{\gamma} \subset \Gamma_{\omega, R^{1/2}/2}^{-3\pi/4, 3\pi/4}$  or  $\gamma$  leaves  $B_{R^{1/2}}$ ; the latter is impossible since  $\text{Length}(\gamma) \leq R^{1/2}/2$ . We conclude that  $\tilde{\gamma} \subset \Gamma_{\omega, R^{1/2}/2}^{-3\pi/4, 3\pi/4}$ . In particular,  $\tilde{\gamma} = \gamma$  and the corollary follows.  $\square$

### II.3. AREA GROWTH OF STABLE SECTORS AND THE PROOF OF THEOREM 0.3

In this section, we show that case (2) in Theorem II.1.2 does not happen and thus Theorem 0.3 follows easily. To do that, we first prove upper and lower bounds for the area of a stable sector over a curve  $\sigma_1$  if the sides  $\gamma_1, \gamma_2$  of the sector are contained in multi-valued graphs  $\Sigma_1, \Sigma_2$ . By [CM8], the number of sheets of each  $\Sigma_i$  grows at least like  $\log^2 \rho$ , giving the lower area bound  $\rho^2 \log^2 \rho$  when the  $\Sigma_i$ 's are disjoint. We use this growing number of sheets to construct a function  $\chi$  with small energy and which vanishes on the sides  $\gamma_1, \gamma_2$ . Inserting  $\chi$  in Lemma II.1.3 gives the upper area bound  $\rho^2(C + \log \log \rho)$  (where  $C = C(\sigma_1)$ ). If  $\rho$  is large depending on  $C$ , then these bounds are contradictory and hence the  $\Sigma_i$ 's cannot be disjoint.

We will use several times that given  $\alpha > 0$ , Proposition II.2.12 of [CM3] gives  $N_g > 0$  so if  $u$  satisfies the minimal surface equation on  $S_{e^{-N_g}, e^{N_g} R}^{-N_g, 2\pi+N_g}$  with  $|\nabla u| \leq 1$ , and  $w < 0$  (where  $w$  is the separation), then  $\rho |\text{Hess}_u| + \rho |\nabla w|/|w| \leq \alpha$  on  $S_{1,R}^{0,2\pi}$ . Theorem 3.36 of [CM7] then yields  $|\nabla u - \nabla u(1,0)| \leq C\alpha$ . We can therefore assume (after rotating so  $\nabla u(1,0) = 0$ ) that

$$|\nabla u| + \rho |\text{Hess}_u| + 4\rho |\nabla w|/|w| + \rho^2 |\text{Hess}_w|/|w| \leq \epsilon < 1/(2\pi). \quad (\text{II.3.1})$$

The bound on  $|\text{Hess}_w|$  follows from the other bounds and standard elliptic theory.

The next lemma shows that an embedded multi-valued minimal graph in a concave cone (intersected with cylindrical shells)  $\mathcal{C}_{\Lambda, R}(h) = \{x \mid (x_3 - h)^2 \leq \Lambda^2 (x_1^2 + x_2^2), 1/4 \leq x_1^2 + x_2^2 \leq R^2\}$  has at least  $\log^2 \rho$  many sheets. Note that the axis of the cone  $\mathcal{C}_{\Lambda, R}(h)$  is the  $x_3$ -axis and the vertex is  $(0, 0, h)$ . We will only need  $\log \rho$  sheets for most of what follows, except for the lower bound for the area given in Corollary II.3.15 below.

**Lemma II.3.2.** Given  $\epsilon > 0$ , there exist  $0 < C_1 < 1$  and  $C_2$  so: Let  $\Sigma \subset \mathcal{C}_{8\pi\epsilon, R}(0)$  with  $\partial\Sigma \subset \partial\mathcal{C}_{8\pi\epsilon, R}(0)$  be a minimal multi-valued graph of  $u$  with  $w < 0$  and  $u(1,0) = 0$ . If the domain of  $u$  contains  $S_{1/2, R}^{-2\pi, 2\pi}$  and  $u$  satisfies (II.3.1), then  $\Sigma$  contains a (multi-valued) graph over

$$\{(\rho, \theta) \mid |\theta| \leq C_1 \log^2 \rho + \pi, 1 \leq \rho \leq R^{3/4}/2\} \quad (\text{II.3.3})$$

with

$$\rho^2 |A|^2 \leq C_2 \rho^{-5/18}. \quad (\text{II.3.4})$$

*Proof.* Corollary 1.14 of [CM8] gives on  $S_{1, R^{3/4}}^{-\pi, \pi}$  that

$$\rho^2 |\text{Hess}_u(\rho, \theta)|^2 \leq C \rho^{-5/18}, \quad (\text{II.3.5})$$

directly giving (II.3.4) for  $|\theta| \leq \pi$ . By corollary 5.7 of [CM8],  $\Sigma$  contains a (multi-valued) graph over  $\{(\rho, \theta) \mid c^2 |\theta| \leq \log^2 \rho, 1 \leq \rho \leq R^{3/4}\}$  with  $|u(\rho, 2\pi n) - u(\rho, 0)| \leq \rho^\epsilon$  for  $n \in \mathbf{Z}$ ,  $2\pi c^2 |n| \leq \log^2 \rho$ . Applying the Harnack inequality and elliptic estimates to the function

$w_n(\rho, \theta) = u(\rho, 2\pi n + \theta) - u(\rho, \theta)$  (cf. (1.17) of [CM8]), we get

$$\rho |\nabla u(\rho, 2\pi n) - \nabla u(\rho, 0)| + \rho^2 |\text{Hess}_u(\rho, 2\pi n) - \text{Hess}_u(\rho, 0)| \leq C' \rho^\epsilon. \quad (\text{II.3.6})$$

Combining (II.3.5) and (II.3.6) then easily gives (II.3.4) in general.  $\square$

We first define a function  $0 \leq \chi \leq 1$  on  $\mathcal{P}$  (the universal cover of  $\mathbf{C} \setminus \{0\}$ ) which is 0 on  $S_{3/4, \infty}^{-\pi, \pi}$ , 1 on  $\{\rho < R^{3/4}/2\} \setminus (S_{1/2, R}^{-2\pi, 2\pi} \cup (\text{II.3.3}))$ , and so  $|\nabla_{\mathcal{P}} \chi|^2$  is of the order  $(\rho \log \rho)^{-2}$  for  $\rho$  large. Namely, set

$$\chi(\rho, \theta) = \begin{cases} 3 - 4\rho & \text{for } 1/2 \leq \rho < 3/4, |\theta| \leq \pi, \\ 1 - (C_1 - |\theta| + \pi)(4\rho - 2)/C_1 & \text{for } 1/2 \leq \rho < 3/4, \pi \leq |\theta| \leq C_1 + \pi, \\ 0 & \text{for } |\theta| \leq \pi, 3/4 \leq \rho, \\ (|\theta| - \pi)/C_1 & \text{for } 3/4 \leq \rho < e, \pi \leq |\theta| \leq C_1 + \pi, \\ (|\theta| - \pi)/(C_1 \log \rho) & \text{for } e \leq \rho, \pi \leq |\theta| \leq C_1 \log \rho + \pi, \\ 1 & \text{otherwise.} \end{cases} \quad (\text{II.3.7})$$

Using (II.3.7), define  $\chi$  on a (multi-valued) graph over a domain containing  $S_{1/2, R}^{-2\pi, 2\pi} \cup (\text{II.3.3})$  in the obvious way. Note that if  $\Sigma$  is as in Lemma II.3.2, then  $1 - \chi$  is one on the central sheet  $\Sigma_{3/4, R}^{-\pi, \pi}$  and vanishes before  $\Sigma$  leaves the cone on the top, bottom, or inside.

**Corollary II.3.8.** Given  $\epsilon > 0$ , there exists  $C_3$  so if  $\Sigma, u$  are as in Lemma II.3.2, then  $\chi = 0$  over  $S_{3/4, R}^{-\pi, \pi}$ ,  $\chi = 1$  on  $\{x_1^2 + x_2^2 \leq R^{3/2}/4\} \cap \partial\Sigma$ , and for  $e < t \leq R^{3/4}/2$

$$\int_{\{\chi < 1, x_1^2 + x_2^2 \leq t^2\}} |A|^2 + \int_{\{x_1^2 + x_2^2 \leq t^2\} \cap \Sigma} |\nabla \chi|^2 \leq C_3 (1 + \log \log t). \quad (\text{II.3.9})$$

*Proof.* Clearly,  $\chi = 0$  over  $S_{3/4, R}^{-\pi, \pi}$ . By Lemma II.3.2,  $\chi = 1$  on  $\{x_1^2 + x_2^2 \leq R^{3/2}/4\} \cap \partial\Sigma$ . To get (II.3.9), first consider  $\chi$  as a function downstairs on  $\mathcal{P}$ . On  $\{\rho \leq e\}$ ,  $|\nabla_{\mathcal{P}} \chi| \leq C_0$  and  $\{|\nabla_{\mathcal{P}} \chi| \neq 0\} \subset \{|\theta| \leq C_1 + \pi, 1/2 \leq \rho\}$ . Similarly, on  $\{e \leq \rho\}$ ,  $|\partial_{\theta} \chi|/\rho \leq 1/(C_1 \rho \log \rho)$  and  $|\partial_{\rho} \chi| \leq 1/(\rho \log \rho)$ , so that  $|\nabla_{\mathcal{P}} \chi|^2 \leq 2(C_1 \rho \log \rho)^{-2}$  and  $\{|\nabla_{\mathcal{P}} \chi| \neq 0\} \subset \{\pi \leq |\theta| \leq C_1 \log \rho + \pi\}$ . Therefore, since  $\Sigma$  is a graph with gradient  $\leq 1$ , it follows easily that

$$\int_{\{x_1^2 + x_2^2 \leq t^2\} \cap \Sigma} |\nabla \chi|^2 \leq C'_0 + \frac{12}{C_1} \int_e^t \frac{ds}{s \log s} = C'_0 + \frac{12 \log \log t}{C_1}. \quad (\text{II.3.10})$$

Similarly, using (II.3.4) gives

$$\int_{\{\chi < 1, x_1^2 + x_2^2 \leq t^2\}} |A|^2 \leq C + 4C_2 \int_e^\infty (\pi + C_1 \log s) s^{-23/18} ds \leq C'. \quad (\text{II.3.11})$$

Finally, combining (II.3.10) and (II.3.11) gives (II.3.9).  $\square$

The next corollary gives upper and lower bounds for the areas of tubular neighborhoods in a  $\Gamma$  which satisfies i)–iii) below. ( $\Gamma_t(\partial)$  is the component of  $B_t \cap \Gamma$  containing  $B_t \cap \partial\Gamma$ .)

i)  $\Gamma \subset B_{2R}$  is a stable embedded minimal surface,  $\partial\Gamma \subset B_{1/4} \cup \partial B_{2R}$ ,  $B_{1/4} \cap \partial\Gamma$  is connected,  $\Gamma_0 \subset \Gamma$  is a disk,  $\partial\Gamma_0 = \gamma_1 \cup \gamma_2 \cup \sigma_1 \cup \sigma_2$  ( $\gamma_i : [0, \ell_i] \rightarrow \Sigma$  is a geodesic),  $\gamma_i(0) \in \sigma_i \subset \Gamma_1(\partial)$ , and  $\gamma_i \perp \sigma_i$ ;

- ii)  $\Sigma_1, \Sigma_2 \subset \Gamma$  are disjoint (multi-valued) graphs over domains containing  $S_{1/2, R}^{-2\pi, 2\pi}$  of  $u_1, u_2$  satisfying (II.3.1),  $w_i < 0$ ,  $\Sigma_i \subset \mathcal{C}_{8\pi\epsilon, R}(u_i(1, 0))$ ,  $\partial\Sigma_i \subset \partial\mathcal{C}_{8\pi\epsilon, R}(u_i(1, 0))$ , and  $\gamma_i \subset (\Sigma_i)_{3/4, R}^{-\pi, \pi}$ ;  
 iii)  $\text{dist}_\Gamma(\gamma_i(t), \Gamma_1(\partial)) = t$  for  $0 \leq t \leq \ell_i$ ,  $\ell_i \geq R - 1$ , and  $\text{dist}_\Gamma(\sigma_2, \Gamma_1(\partial)) \geq R - 1$ .

We first show that intrinsic and extrinsic distances to  $\sigma_1$  are roughly equivalent:

**Lemma II.3.12.** There exists  $C_d > 1$  so if i)–iii) hold and  $R > C_d$ , then  $B_{R/C_d} \cap \sigma_2 = \emptyset$  and  $B_{t/C_d} \cap \Gamma_0 \subset \mathcal{T}_t(\sigma_1, \Gamma_0)$  for  $C_d < t < R$ .

*Proof.* Both of these assertions follow easily from stability together with the assumption that  $\Gamma$  contains multi-valued graphs. Namely, suppose that either one failed. It follows easily that there exists a point in  $\Gamma$  which is extrinsically much closer to the origin than its intrinsic distance to the inner boundary of  $\Gamma$ . This easily implies by stability that  $\Gamma$  contains a large almost flat graph over a disk centered at the origin which easily contradicts that  $\Gamma$  contains multi-valued graphs since these would be forced to spiral into the almost flat graph. We will now make this argument precise.

Fix  $C_d > 1$  to be chosen. We show first that  $B_t \cap \Gamma \subset \mathcal{T}_{C_d t}(B_{1/4} \cap \partial\Gamma)$  for  $1 < t < R/C_d$ . Suppose that  $y \in B_{R/C_d} \cap \Gamma$ . Fix  $C > 2$  and  $\delta > 0$  to be chosen. Since  $\Gamma$  is stable, [Sc], [CM2] give  $C'_d = C'_d(C, \delta)$  so that if  $\text{dist}_\Gamma(y, \partial\Gamma) > C'_d(1 + |y|)$ , then  $\mathcal{B}_{C'_d(1+|y|)}(y)$  contains a graph  $\Gamma_y$  with gradient  $\leq \delta$  over a disk  $B_{C(1+|y|)}(y) \cap P_y$ , where  $P_y \subset \mathbf{R}^3$  is the plane tangent to  $\Gamma$  at  $y$ . Since  $\Gamma$  is embedded (and since  $\Gamma$  contains a multi-valued graph  $\Sigma_1$  around  $\gamma_1$  with  $\gamma_1(0) \in B_1$ ), we can choose  $C, \delta$  so  $\Gamma$  would then be forced to spiral into  $\Gamma_y$ . This is impossible since  $\Gamma$  is compact. Since  $\partial\Gamma \subset \Gamma_1(\partial) \cup \partial B_{2R}$ , it follows that  $B_t \cap \Gamma \subset \mathcal{T}_{2C'_d t}(B_{1/4} \cap \partial\Gamma)$  for  $1 < t < R/C_d$ . Combining this and iii) gives  $B_{(R-1)/(2C'_d)} \cap \sigma_2 = \emptyset$ .

Suppose that  $y \in B_{R/C_d} \cap \Gamma_0$  so (by the first part) we get  $y' \in \partial\Gamma_0$  with

$$\text{dist}_{\Gamma_0}(y, y') + \text{dist}_\Gamma(y', B_{1/4} \cap \partial\Gamma) \leq C'_d(1 + |y|) < R. \quad (\text{II.3.13})$$

In particular,  $y' \in \sigma_1 \cup \gamma_1 \cup \gamma_2$ . Since  $\text{dist}_\Gamma(\gamma_i(t), \Gamma_1(\partial)) = t$ , we get

$$\text{dist}_{\sigma_1 \cup \gamma_i}(y', \sigma_1) \leq C'_d(1 + |y|), \quad (\text{II.3.14})$$

so  $\text{dist}_{\Gamma_0}(y, \sigma_1) \leq 2C'_d(1 + |y|)$  and the lemma follows.  $\square$

**Corollary II.3.15.** Given  $\epsilon, C_I > 0$ , there exists  $C_4 > 0$  so if i)–iii) hold and  $R^{3/4} > 12C_d$ , then for  $e < t \leq R^{3/4}/4 - 1$

$$C_4 \log^2 t \leq t^{-2} \text{Area}(\mathcal{T}_t(\sigma_1, \Gamma_0)) \leq \left( 1 + \int_{\mathcal{T}_{C_I}(\sigma_1, \Gamma_0)} (1 + |A|^2) + \int_{\sigma_1} (1 + |k_g|) + \log \log t \right) / C_4. \quad (\text{II.3.16})$$

*Proof.* Since  $\sigma_1 \subset \Gamma_1(\partial)$ , i) and iii) imply (A) with  $C_0 = 0$ , (C), and (D) with  $\ell = R - 1$ . Using Corollary II.3.8 on  $\Sigma_1, \Sigma_2$ , we can define  $\chi$  on  $\{x_1^2 + x_2^2 \leq R^{3/2}/4\} \cap \Gamma$  which vanishes on  $\gamma_1, \gamma_2$ , is one on  $\{x_1^2 + x_2^2 \leq R^{3/2}/4\} \cap \Gamma \setminus (\Sigma_1 \cup \Sigma_2)$ , and satisfies (II.3.9) (with double the constant). Since  $\mathcal{T}_t(\sigma_1, \Gamma_0) \subset \{x_1^2 + x_2^2 \leq R^{3/2}/4\}$ , inserting (II.3.9) into Lemma II.1.3 (and scaling so  $C_I \rightarrow 1$ ) gives the second inequality in (II.3.16).

By Lemma II.3.2,  $\Sigma_1, \Sigma_2$  each contain a (multi-valued) graph over (II.3.3). Suppose now that  $e < t < R^{3/4}/(4C_d)$ . By Lemma II.3.12,  $\{1 < x_1^2 + x_2^2 \leq t^2\} \cap \Gamma \subset B_{2t} \cap \Gamma \subset \mathcal{T}_{2C_d t}(\sigma_1, \Gamma)$  and  $B_{2t} \cap \sigma_2 = \emptyset$  (by iii)). Since  $\sigma_1 \subset B_1$ ,  $\gamma_i \subset (\Sigma_i)_{3/4, R}^{-\pi, \pi}$ , and  $\Sigma_1 \cap \Sigma_2 = \emptyset$ , it then follows easily that  $\mathcal{T}_{2C_d t}(\sigma_1, \Gamma_0)$  contains one component of  $\{1 < x_1^2 + x_2^2 \leq t^2\} \cap \Sigma_1 \setminus (\Sigma_1)_{1, t}^{-\pi, \pi}$ . The first inequality in (II.3.16) follows immediately (after possibly decreasing  $C_4 > 0$ ).  $\square$

*Proof.* (of Theorem 0.3). Rescale so that  $r_0 = 1$ . Set  $\hat{\Gamma} = \Gamma \setminus \Gamma_1(\partial)$  so (since  $\Gamma$  is topologically an annulus)  $\partial\hat{\Gamma} = \sigma \cup \hat{\sigma}$  where  $\sigma \subset \partial B_1$ ,  $\hat{\sigma} \subset \partial B_R$  are the two connected components of  $\partial\hat{\Gamma}$ , and  $\partial_{\mathbf{n}}|x| \geq 0$  on  $\sigma$  (where  $\mathbf{n}$  is the inward normal to  $\partial\hat{\Gamma}$ ).

By Theorem II.1.2 we need only prove that (2) does not happen for  $\hat{\Gamma}$ . Suppose it does; we will obtain a contradiction. The key point will be to find two oppositely oriented multi-valued graphs in  $\Gamma$  which have fixed bounded distance between them and then apply Corollary II.3.15 for  $t$  sufficiently large to get a contradiction.

Fix (ordered) points  $z_1, \dots, z_m \in \sigma$  so  $\sigma \setminus \{z_1, \dots, z_m\}$  has components  $\{\sigma_{z_1}, \dots, \sigma_{z_m}\}$  where  $\partial\sigma_{z_i} = \{z_i, z_{i+1}\}$  (set  $z_{m+1} = z_1$ ) and  $\text{Length}(\sigma_{z_i}) \leq 1$ . By Theorem II.1.2 (and the discussion surrounding (II.3.1)),  $\Gamma$  contains 3-valued graphs  $\Sigma_{z_i}$  of  $u_{z_i}$  satisfying (II.3.1) over  $D_{R/\omega} \setminus D_\omega$  (after a rotation of  $\mathbf{R}^3$ ; a priori this rotation may depend on  $z_i$ ) and with  $\text{dist}_{\hat{\Gamma}}(z_i, (\Sigma_{z_i})_{\omega, \omega}^{0,0}) < d_0$ . Combining this with Corollary II.2.15, we get 3-valued graphs  $\{\Sigma_{z_i}\}$ , geodesics  $\gamma_{z_i} : [0, \ell_{z_i}] \rightarrow (\Sigma_{z_i})_{\omega, R^{1/2}}^{-\pi, \pi}$  with  $\gamma_{z_i}(0) \in \partial B_{\lambda\omega}$ ,  $\text{dist}_{\Gamma}(\gamma_{z_i}(t), \Gamma_{\lambda\omega}(\partial)) = t$  for  $0 \leq t \leq \ell_{z_i}$ , and  $\gamma_i(\ell_{z_i}) \subset \Gamma \setminus B_{R^{1/2}/3}$ . After possibly increasing  $\lambda$ , we can assume that  $\lambda\omega > 2d_0 + 2$ . Hence, the curves in  $\hat{\Gamma}$  from  $z_i$  to  $(\Sigma_{z_i})_{\omega, \omega}^{0,0}$  given by Theorem II.1.2 are contained in  $B_{\lambda\omega/2}$ . Therefore, since  $(\Sigma_{z_i})_{\omega, \lambda\omega}^{-3\pi, 3\pi}$  is a graph, we can choose curves  $\eta_{z_i} \subset \Gamma_{\lambda\omega}(\partial)$  from  $\gamma_{z_i}(0)$  to  $z_i$  with length  $\leq 2\lambda\omega + 4\pi\omega$  and so  $\eta_{z_i} \setminus B_{\lambda\omega/2}$  is simple with  $\int_{\eta_{z_i} \setminus B_{\lambda\omega/2}} |k_g| \leq C$ .

It follows immediately from embeddedness that the  $\Sigma_{z_i}$ 's are graphs over a common plane. From the gradient estimate (which applies because of estimates for stable surfaces of [Sc], [CM2]), each component of  $\Gamma$  intersected with a concave cone is also a multi-valued graph. Since  $\partial B_{\lambda\omega} \cap \partial\Gamma_{\lambda\omega}(\partial)$  is a closed curve, it must pass between the sheets of each  $\Sigma_{z_i}$ . It is now easy to see that each  $\Sigma_{z_i}$  contains an oppositely oriented multi-valued graph  $\hat{\Sigma}_{z_i}$  between its sheets (i.e.,  $\mathbf{n}_{\Gamma}$  points in almost opposite directions on  $\Sigma_{z_i}$  and  $\hat{\Sigma}_{z_i}$ ). Furthermore, since Lemma II.3.12 bounds the distance in  $\hat{\Gamma}$  from  $\hat{\Sigma}_{z_i}$  to  $\sigma$ , we can assume that two of the  $\Sigma_{z_i}$ 's are oppositely oriented. We can therefore choose two consecutive 3-valued graphs,  $\Sigma_{z_j}, \Sigma_{z_{j+1}}$ , which are oppositely oriented; rename these  $\Sigma_1, \Sigma_2$  (and similarly the corresponding  $\gamma_1, \gamma_2, \ell_1, \ell_2$ ).

By replacing  $B_{\lambda\omega/2} \cap (\sigma_{z_j} \cup \eta_{z_j} \cup \eta_{z_{j+1}})$  with a broken geodesic and finding a simple subcurve as in Lemma II.1.11, we get a simple curve  $\sigma_1 \subset \Gamma_{\lambda\omega}(\partial) \setminus \Gamma_{7/8}(\partial)$  from  $\gamma_1(0)$  to  $\gamma_2(0)$  with

$$\int_{\sigma_1} (1 + |k_g|) \leq C_a. \quad (\text{II.3.17})$$

Furthermore, since  $\sigma_1 \subset \Gamma_{\lambda\omega}(\partial)$ ,  $\text{dist}_{\Gamma}(\gamma_i(t), \sigma_1) = t$  for  $0 \leq t \leq \ell_i$ . Let  $\Gamma_0$  be the component of  $\Gamma_{R^{1/2}/3}(\partial) \setminus (\sigma_1 \cup \gamma_1 \cup \gamma_2)$  which does not contain  $\Gamma_{7/8}(\partial)$ ; set  $\sigma_2 = \partial\Gamma_0 \setminus (\sigma_1 \cup \gamma_1 \cup \gamma_2)$ . It follows that  $\Gamma_0$  is a disk and  $\text{dist}_{\Gamma}(\Gamma_0, \partial\Gamma) \geq 5/8$ . Since  $(\Sigma_{z_i})_{\omega, \lambda\omega}^{-3\pi, 3\pi}$  is a graph, we can perturb  $\sigma_1$  near  $\gamma_1(0), \gamma_2(0)$  to arrange that  $\sigma_1 \perp \gamma_1$  and  $\sigma_1 \perp \gamma_2$  and so  $\sigma_1$  still satisfies (II.3.17) with a slightly larger constant  $C_a$ . Combining (II.3.17) and estimates for stable surfaces of [Sc], [CM2], we get

$$\int_{\mathcal{T}_{1/8}(\sigma_1, \Gamma_0)} (1 + |A|^2) + \int_{\sigma_1} (1 + |k_g|) \leq C_b. \quad (\text{II.3.18})$$

Hence, (after rescaling)  $\Gamma_0, \Gamma, \Sigma_1, \Sigma_2, \gamma_1, \gamma_2, \sigma_1$  satisfy i)–iii). To get ii), we use [Sc], [CM2] and the gradient estimate to extend  $\Sigma_1, \Sigma_2$  as multi-valued graphs inside the cones  $\mathcal{C}_{8\pi\epsilon, R^{1/2}/4}(u_i(1, 0))$ ; the opposite orientation guarantees that  $\Sigma_1 \cap \Sigma_2 = \emptyset$ . Corollary II.3.15 and (II.3.18) give

for  $C_5 < t < R^{3/8}/C_5$

$$C_4 \log^2 t \leq t^{-2} \text{Area}(\mathcal{T}_t(\sigma_1, \Gamma_0)) \leq (1 + C_b + \log \log t) / C_4. \quad (\text{II.3.19})$$

This gives the desired contradiction for  $R$  large, completing the proof.  $\square$

### Part III. Nearby points with large curvature

In this part, we extend Theorem 0.3 (proven for stable surfaces) to surfaces with *extrinsic* quadratic curvature decay  $|A|^2 \leq C|x|^{-2}$ . As mentioned in the introduction, this extension is needed in both [CM5] and [CM6]. In [CM5] it is used for disks to get points of large curvature nearby and on each side of a given point with large curvature (in particular it is used to show that such points are not extrinsically isolated).

Stability was used in the proof of Theorem 0.3 for two purposes: (a) To show *intrinsic* quadratic curvature decay. (b) To bound the total curvature using the stability inequality. To get the extension to the extrinsic quadratic curvature decay case, we will deal with (a) and (b) separately in the next two sections. To get (a), we relate extrinsic and intrinsic distances (i.e., we show a “chord-arc” property). For (b), we follow section 2 of [CM4] to decompose a surface with quadratic curvature decay into disjoint almost stable subdomains and a “remainder” with quadratic area growth.

For applications of the results of this part in [CM5],  $\Sigma$  will be a disk and hence  $\partial\Sigma_{0,t}$  is connected for all  $t$  (here, and elsewhere, if  $0 \in \Sigma$ , then  $\Sigma_{0,t}$  denotes the component of  $B_t \cap \Sigma$  containing 0). However, in [CM6], when we apply the results here to deal with the first possibility in “4.” of Theorem 0.1 (i.e., the analog of the genus one helicoid),  $\Sigma$  is no longer a disk but  $\partial\Sigma$  is still connected (which is assumed in many of the results below).

#### III.1. RELATING INTRINSIC AND EXTRINSIC DISTANCES

In this section,  $0 \in \Sigma \subset B_R$  is an embedded minimal surface,  $\partial\Sigma \subset \partial B_R$ ,  $|A|^2 \leq C_1^2|x|^{-2}$  on  $\Sigma \setminus B_1$ , and  $\partial\Sigma_{0,t}$  is connected for  $1 \leq t \leq R$ .

The next lemma shows that only one component of  $B_{C_b} \cap \Sigma$  intersects  $B_2$ . The second lemma bounds the radius of the intrinsic tubular neighborhood of  $B_2 \cap \Sigma$  containing this component. Combining these iteratively (on decreasing scales) in Corollary III.1.5 gives the “chord-arc” property needed to establish (a).

**Lemma III.1.1.** Given  $C_1$ , there exists  $C_b$  so if  $\Sigma_{0,1}$  is not a graph, then  $B_2 \cap \Sigma \subset \Sigma_{0,C_b}$ .

*Proof.* Suppose that  $\Sigma_1, \Sigma_2$  are disjoint components of  $B_{C_b} \cap \Sigma$  with  $B_2 \cap \Sigma_i \neq \emptyset$ . It follows that there is a component  $\Omega$  of  $B_{C_b} \setminus \Sigma$  and a segment  $\eta \subset B_2 \setminus \Sigma$  so that  $\partial\Sigma_{0,C_b}$  is linked with  $\eta$  in  $\Omega$  (cf. lemma 2.1 in [CM9]). Since  $\Omega$  is mean convex, we can solve the Plateau problem as in [MeYa2] to get a stable minimal surface  $\Gamma \subset \Omega$  with  $\partial\Gamma = \partial\Sigma_{0,C_b}$ . The linking implies that  $B_2 \cap \Gamma \neq \emptyset$ . The curvature estimates of [Sc], [CM2] then give a graph  $\Gamma_0 \subset \Gamma$  of a function  $u_0$  over  $D_{C_b/C}$  (after a rotation) with  $|u_0(z)| \leq |z|$ . By corollary 1.14 of [CM8] (applied with  $w = 0$ ), we can assume that on  $D_{C_b^{1/2}/C}$

$$|\nabla u_0|(z) \leq C' |z|^{-5/12}. \quad (\text{III.1.2})$$

In particular,  $\Gamma_0$  is close to a horizontal plane. The lemma now follows from an argument used in [CM9] (see also [CM10]) which we now outline:  $\Sigma$  intersects a narrow cone about  $\Gamma_0$ , then contains a long chain of graphical balls (by the gradient estimate), and must then

either spiral indefinitely or close up as a graph. Namely, for  $t < C_b^{1/2}/C$ ,  $\Sigma_{0,t}$  sits on one side of  $\Gamma_0$ . However, by lemma 2.4 of [CM9] (for  $t > C'$ ),  $\partial\Sigma_{0,t}$  contains a ‘‘low point’’  $y_0$  (i.e.,  $|x_3(y_0)| \leq \delta t$  with  $\delta > 0$  small). The gradient estimate (since  $|A|^2 \leq C_1^2|x|^{-2}$  on  $\Sigma \setminus B_1$ ) gives a long chain of balls  $\mathcal{B}_{Ct}(y_i)$  with  $y_i \in \partial\Sigma_{0,t} \cap \{|x_3| \leq C' \delta t\}$  and which is a (possibly multi-valued) graph. Since  $\partial\Sigma_{0,t}$  cannot spiral forever, this graph closes up. By Rado’s theorem (note that no assumption on the topology is needed for this application of Rado’s theorem; cf. the proof of theorem 1.22 in [CM4]),  $\Sigma_{0,t}$  is itself a graph, giving the lemma.  $\square$

**Lemma III.1.3.** Given  $C_1, C_b$ , there exists  $C_c$  so if  $R > C_c$ , then for all  $y \in \Sigma_{0,C_b}$

$$\text{dist}_\Sigma(y, B_1 \cap \Sigma) \leq C_c. \quad (\text{III.1.4})$$

*Proof.* Let  $\tilde{\Sigma}$  be the universal cover of  $\Sigma$  and  $\tilde{\Pi} : \tilde{\Sigma} \rightarrow \Sigma$  the covering map. With the definition of  $\delta$ -stable as in section 2 of [CM4], the argument of [CM2] (i.e., curvature estimates for 1/2-stable surfaces) gives  $C > 10$  so if  $\mathcal{B}_{CC_b/2}(\tilde{z}) \subset \tilde{\Sigma}$  is 1/2-stable and  $\tilde{\Pi}(\tilde{z}) = z$ , then  $\tilde{\Pi} : \mathcal{B}_{5C_b}(\tilde{z}) \rightarrow \mathcal{B}_{5C_b}(z)$  is one-to-one and  $\mathcal{B}_{5C_b}(z)$  is a graph with  $B_{4C_b}(z) \cap \partial\mathcal{B}_{5C_b}(z) = \emptyset$ . Corollary 2.13 in [CM4] gives  $\epsilon = \epsilon(C, C_1, C_b) > 0$  so that if  $|z_1 - z_2| < \epsilon$  and  $|A|^2 \leq C_1^2$  on (the disjoint balls)  $\mathcal{B}_{CC_b}(z_i)$ , then each  $\mathcal{B}_{CC_b/2}(\tilde{z}_i) \subset \tilde{\Sigma}$  is 1/2-stable where  $\tilde{\Pi}(\tilde{z}_i) = z_i$ .

We claim that there exists  $n$  so  $B_1 \cap \mathcal{B}_{(2n+1)C_b}(y) \neq \emptyset$ . Suppose not; we get a curve  $\sigma \subset \Sigma_{0,C_b} \setminus \mathcal{T}_{CC_b}(B_1 \cap \Sigma)$  from  $y$  to  $\partial\mathcal{B}_{2nCC_b}(y)$ . For  $i = 1, \dots, n$ , fix points  $z_i \in \partial\mathcal{B}_{2iCC_b}(y) \cap \sigma$ . The intrinsic balls  $\mathcal{B}_{CC_b}(z_i) \subset \Sigma \setminus B_1$  are disjoint, have centers in  $B_{C_b} \subset \mathbf{R}^3$ , and  $\sup_{\mathcal{B}_{CC_b}(z_i)} |A|^2 \leq C_1^2$ . Hence, there exist  $i_1, i_2$  with  $0 < |z_{i_1} - z_{i_2}| < C' C_b n^{-1/3} < \epsilon$ , and, by corollary 2.13 in [CM4], each  $\mathcal{B}_{CC_b/2}(\tilde{z}_{i_j}) \subset \tilde{\Sigma}$  is 1/2-stable where  $\tilde{\Pi}(\tilde{z}_{i_j}) = z_{i_j}$ . By [CM2], each  $\mathcal{B}_{5C_b}(z_{i_j})$  is a graph with  $B_{4C_b}(z_{i_j}) \cap \partial\mathcal{B}_{5C_b}(z_{i_j}) = \emptyset$ . In particular,  $B_{C_b} \cap \partial\mathcal{B}_{5C_b}(z_{i_j}) = \emptyset$ . This contradicts that  $\sigma \subset B_{C_b}$  connects  $z_{i_j}$  to  $\partial\mathcal{B}_{CC_b}(z_{i_j})$ .  $\square$

**Corollary III.1.5.** Given  $C_1$ , there exists  $C_c$  so if  $\Sigma_{0,1}$  is not a graph,  $y \in B_{R/C_c} \cap \Sigma$ , then

$$\text{dist}_\Sigma(y, B_1 \cap \Sigma) \leq 2 C_c |y|. \quad (\text{III.1.6})$$

*Proof.* Suppose  $y \in B_{2^n} \setminus B_{2^{n-1}}$ . By Lemma III.1.1,  $y \in \Sigma_{0,C_b 2^{n-1}}$  where  $C_b = C_b(C_1)$ . Set  $y_n = y$ . Lemma III.1.3 gives  $y_{n-1} \in B_{2^{n-1}} \cap \Sigma$  with  $\text{dist}_\Sigma(y_n, y_{n-1}) \leq C_c 2^{n-1}$ .

We can now repeat the argument. Namely, by Lemma III.1.1,  $y_{n-1} \in \Sigma_{0,C_b 2^{n-2}}$  and then Lemma III.1.3 gives  $y_{n-2} \in B_{2^{n-2}} \cap \Sigma$  with  $\text{dist}_\Sigma(y_{n-1}, y_{n-2}) \leq C_c 2^{n-2}$ , etc. After  $n$  steps, we get  $y_0 \in B_1 \cap \Sigma$  with

$$\text{dist}_\Sigma(y, y_0) \leq \sum_{i=1}^n \text{dist}_\Sigma(y_i, y_{i-1}) \leq \sum_{i=1}^n C_c 2^{i-1} \leq 2 C_c |y|. \quad (\text{III.1.7})$$

$\square$

## III.2. A DECOMPOSITION FROM [CM4]

In lemma 2.15 of [CM4], we decomposed an embedded minimal surface in a ball with bounded curvature into disjoint almost stable subdomains and a remainder with bounded area. The same argument gives:

**Lemma III.2.1.** Given  $C_1$ , there exists  $C_d$  so: If  $\Sigma \subset B_{2R}$  is an embedded minimal surface with  $\partial\Sigma \subset \partial B_{2R} \cup B_{1/2}$ , and  $|A|^2 \leq C_1^2 |x|^{-2}$ , then there exist disjoint  $1/2$ -stable subdomains  $\Omega_j \subset \Sigma$  and a function  $0 \leq \psi \leq 1$  on  $\Sigma$  which vanishes on  $(B_R \setminus B_1) \cap \Sigma \setminus (\cup_j \Omega_j)$  so

$$\text{Area}(\{x \in (B_R \setminus B_1) \cap \Sigma \mid \psi(x) < 1\}) \leq C_d R^2, \quad (\text{III.2.2})$$

$$\int_{B_R \cap \Sigma} |\nabla \psi|^2 \leq C_d \log R. \quad (\text{III.2.3})$$

In the proof of Theorem 0.5 in the next section, Lemma III.2.1 will be used to extend the area bounds for stable surfaces proven in Sections II.1 and II.3 (specifically those in Lemma II.1.3, Proposition II.1.20, and Corollary II.3.15) to minimal surfaces with  $|A|^2 \leq C_1^2 |x|^{-2}$ . This is very similar to how lemma 2.15 of [CM4] was used in lemma 3.1 of [CM4].

By Lemma III.2.1,  $\int_{B_R \cap \Sigma} |\nabla \psi|^2 + \int_{B_R \cap \{\psi < 1\}} |A|^2$  grows (in  $R$ ) at most like  $\log R$ . We use this below in the  $1/2$ -stability inequality to get the total curvature bound needed for (b). This is used in the proof of Theorem III.3.1.

### III.3. THEOREM 0.5 AND A GENERALIZATION

As already mentioned, stability was used in the proof of Theorem 0.3 to establish (a) and (b) in the introduction to Part III; these were extended in the two previous sections to surfaces with a quadratic curvature bound. In [CM5] we will need the contrapositive of Theorem 0.5, i.e., we will need to find points where the quadratic bound fails. In fact, what we will really need is to find points on “each side” of a multi-valued graph where this fails; this is the following theorem:

(Here  $u_1(r_0, 2\pi) < u_2(r_0, 0) < u_1(r_0, 0)$  just says that the two graphs spiral together, one inside the other; cf. theorem 0.6 in [CM8].)

**Theorem III.3.1.** Given  $C_1$ , there exists  $C_2$  so: Let  $0 \in \Sigma \subset B_{2C_2 r_0}$  be an embedded minimal surface with connected  $\partial\Sigma \subset \partial B_{2C_2 r_0}$  and  $\text{gen}(\Sigma_{0,r_0}) = \text{gen}(\Sigma)$ . Suppose  $\Sigma_1, \Sigma_2 \subset \Sigma \cap \{x_3^2 \leq (x_1^2 + x_2^2)\}$  are (multi-valued) graphs of  $u_i$  satisfying (II.3.1) on  $S_{r_0, C_2 r_0}^{-2\pi, 2\pi}$ ,  $u_1(r_0, 2\pi) < u_2(r_0, 0) < u_1(r_0, 0)$ , and  $\nu \subset \partial\Sigma_{0,2r_0}$  a curve from  $\Sigma_1$  to  $\Sigma_2$ . If  $\Sigma_0$  is the component of  $\Sigma_{0, C_2 r_0} \setminus (\Sigma_1 \cup \Sigma_2 \cup \nu)$  which does not contain  $\Sigma_{0,r_0}$ , then

$$\sup_{x \in \Sigma_0 \setminus B_{4r_0}} |x|^2 |A|^2(x) \geq 4C_1^2. \quad (\text{III.3.2})$$

*Proof.* Suppose that (III.3.2) fails for some  $C_1$ ; as in the proof of Theorem 0.3, we will show contradictory upper and lower bounds for the area growth for  $C_2$  sufficiently large.

Note that for  $r_0 \leq s \leq 2C_2 r_0$ , it follows from the maximum principle (since  $\Sigma$  is minimal) and Corollary I.0.11 that  $\partial\Sigma_{0,s}$  is connected and  $\Sigma \setminus \Sigma_{0,s}$  is an annulus.

Note also that the gradient estimate (which applies because of the curvature bound) allows us to extend each  $\Sigma_i$  (inside  $\Sigma_0$ ) as a graph of  $u_i$  over  $\partial D_\rho$  as long as  $|u_i(\rho, \theta) - u_i(\rho, [\theta])| \leq C_g \rho$ , where  $\theta - [\theta] \in 2\pi\mathbf{Z}$  and  $0 \leq [\theta] \leq 2\pi$ . By corollary 1.14 of [CM8], the curvature of  $\Sigma_i$  decays faster than quadratically. Combining these (and increasing the inner radius), we can assume that each  $\Sigma_i$  extends (inside  $\Sigma_0$ ) as a graph until it leaves a cone  $\{x_3^2 \leq \Lambda^2(x_1^2 + x_2^2)\}$  for some small  $\Lambda > 0$ . Moreover, these extended multi-valued graphs must stay disjoint since  $u_1(r_0, 2\pi) < u_2(r_0, 0) < u_1(r_0, 0)$ .

We next choose the inner boundary curve where we argue as in Theorem 0.3. By Lemma III.1.1,  $B_{4r_0} \cap \Sigma \subset \Sigma_{0, 2C_b r_0}$ . In particular,  $\partial\Sigma_{0, 2C_b r_0}$  separates  $B_{4r_0} \cap \Sigma$  from  $\partial\Sigma$ . We can

therefore replace  $\nu$  with a segment of  $\partial\Sigma_{0,2C_b r_0}$  from  $\Sigma_1$  to  $\Sigma_2$  so (for the new  $\Sigma_0$ )

$$\sup_{x \in \Sigma_0} |x|^2 |A|^2(x) \leq 4 \bar{C}_1^2. \quad (\text{III.3.3})$$

By Corollary III.1.5 (the ‘‘chord-arc’’ property), intrinsic and extrinsic distances to  $B_{4r_0} \cap \Sigma$  are compatible. Hence, we get

$$\sup_{x \in \Sigma_0} \text{dist}_\Sigma^2(x, B_{4r_0} \cap \Sigma) |A|^2(x) \leq C_3. \quad (\text{III.3.4})$$

The proof of Theorem 0.3 now applies with two changes (and the minor modifications which result):

(a’) The curvature estimates for stable surfaces of [Sc], [CM2] are replaced with (III.3.4).

(b’) The total curvature bound from the stability inequality in (II.1.6) is replaced with the bound using Lemma III.2.1 and the 1/2-stability inequality (cf. lemma 3.1 of [CM4]).

Namely, using (a’) and (b’), the proof of Theorem II.1.2 extends from stable surfaces to surfaces satisfying (III.3.4) (with (b’) being used in Lemma II.1.3 and Proposition II.1.20 exactly as in [CM4]). It follows that each  $z$  in (the new)  $\nu$  is a fixed bounded distance from a multi-valued graph (either  $\Sigma_1, \Sigma_2$  or a new multi-valued graph in between). Hence, as in the proof of Theorem 0.3, we can choose two consecutive multi-valued graphs which are oppositely oriented; let  $\sigma_1$  be the curve connecting these. Next, (b’) contributes a new  $C_4 t^2 \log t$  term to the upper bound for the area of a sector  $\mathcal{T}_t(\sigma_1)$  in the upper bound for the area in Corollary II.3.15 where  $C_4$  does not depend on  $\sigma_1$  (see the last paragraph of Section III.2). However, since the lower bound for the area is on the order of  $t^2 \log^2 t$ , we get the desired contradiction as before.  $\square$

In [CM5], we will use the special case of Theorem III.3.1 where  $\Sigma$  is a disk:

**Corollary III.3.5.** Given  $C_1$ , there exists  $C_2$  so: Let  $0 \in \Sigma \subset B_{2C_2 r_0}$  be an embedded minimal disk. Suppose  $\Sigma_1, \Sigma_2 \subset \Sigma \cap \{x_3^2 \leq (x_1^2 + x_2^2)\}$  are graphs of  $u_i$  satisfying (II.3.1) on  $S_{r_0, C_2 r_0}^{-2\pi, 2\pi}$ ,  $u_1(r_0, 2\pi) < u_2(r_0, 0) < u_1(r_0, 0)$ , and  $\nu \subset \partial\Sigma_{0,2r_0}$  a curve from  $\Sigma_1$  to  $\Sigma_2$ . Let  $\Sigma_0$  be the component of  $\Sigma_{0, C_2 r_0} \setminus (\Sigma_1 \cup \Sigma_2 \cup \nu)$  which does not contain  $\Sigma_{0, r_0}$ . Suppose either  $\partial\Sigma \subset \partial B_{2C_2 r_0}$  or  $\Sigma$  is stable and  $\Sigma_0$  does not intersect  $\partial\Sigma$ . Then

$$\sup_{x \in \Sigma_0 \setminus B_{4r_0}} |x|^2 |A|^2(x) \geq 4 C_1^2. \quad (\text{III.3.6})$$

*Proof.* Since  $\Sigma$  is a disk,  $\partial\Sigma$  is connected and  $\text{gen}(\Sigma_{0, r_0}) = \text{gen}(\Sigma) = 0$ . Hence, Theorem III.3.1 gives the corollary when  $\partial\Sigma \subset \partial B_{2C_2 r_0}$ .

When  $\Sigma$  is stable and  $\Sigma_0$  does not intersect  $\partial\Sigma$ , then  $\Sigma_1, \Sigma_2$  each extend inside cones in at least one direction as multi-valued graphs. This gives essentially half of the multi-valued graphs  $\Sigma_1, \Sigma_2$  used in Section II.3 which is all that is needed in the proof of Theorem 0.3. The corollary now follows easily from the proof of Theorem 0.3 (with  $\Sigma_1, \Sigma_2$  causing the same modifications as in Theorem III.3.1).  $\square$

Note that if  $C_1$  is large, then (III.3.6) would contradict the curvature estimate for stable surfaces of [Sc], [CM2]. In [CM5], we will apply Corollary III.3.5 in this way, showing that such a stable  $\Sigma$  does not exist.

In [CM5], we will also use the other case of Corollary III.3.5, where  $\Sigma$  is not assumed to be stable, to get points of large curvature ‘‘metrically’’ on each side of the multi-valued graph  $\Sigma_1$ . Namely, note first that the curve  $\partial\Sigma_{0,2r_0} \setminus \nu$  in Corollary III.3.5 has the same properties

as  $\nu$ . In [CM5],  $\nu$  (and hence also  $\Sigma_0$ ) will be on one side of  $\Sigma_1, \Sigma_2$  while  $\partial\Sigma_{0,2r_0} \setminus \nu$  is on the other. Applying Corollary III.3.5 to each of these will give points of large curvature “topologically” on each side of  $\Sigma_1, \Sigma_2$ .

In fact, we will see in [CM5] that if an embedded minimal disk  $\Sigma$  contains one multi-valued graph  $\Sigma_1$ , then it will contain a second multi-valued graph  $\Sigma_2$  which spirals together with  $\Sigma_1$  (“the other half”). We will also see there that  $\partial\Sigma_{0,Cr_0} \setminus (\Sigma_1 \cup \Sigma_2)$  has exactly two components  $\nu_{\pm}$ ; it follows easily that we can assume  $\nu_+$  is above and  $\nu_-$  is below  $\Sigma_1$ . Applying Corollary III.3.5 to both  $\nu_{\pm}$  will give points of large curvature “metrically” on each side of  $\Sigma_1$ .

*Proof.* (of Theorem 0.5). It suffices to show that if  $\text{Area}(\Sigma_{0,r_0}) > C_3 r_0^2$ , then (0.6) fails.

Note that for  $r_0 \leq s \leq R$ , it follows from the maximum principle (since  $\Sigma$  is minimal) and Corollary I.0.11 that  $\partial\Sigma_{0,s}$  is connected and  $\Sigma \setminus \Sigma_{0,s}$  is an annulus.

The proof is now virtually identical to the proof of Theorem III.3.1 except that it simplifies since we no longer keep track of the two sides and (1) in (analog of) Theorem II.1.2 becomes  $\text{Area}(\Sigma_{0,r_0}) \leq C'_3 r_0^2$ .  $\square$

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COURANT INSTITUTE OF MATHEMATICAL SCIENCES AND PRINCETON UNIVERSITY, 251 MERCER STREET,  
NEW YORK, NY 10012 AND FINE HALL, WASHINGTON RD., PRINCETON, NJ 08544-1000

DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, 3400 N. CHARLES ST., BALTIMORE,  
MD 21218

*E-mail address:* `colding@cims.nyu.edu` and `minicozz@math.jhu.edu`