Embedded minimal surfaces.

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Joint work with Tobias H. Colding of MIT and Courant.

Some of this is described in our paper
"Shapes of embedded minimal surfaces"
Proceedings of the National Academy of Sciences,
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Surface $\Sigma \subset \mathbf{R}^3$ is *minimal* if it is a critical point for area. Equivalently:

- Mean curvature $H =$ Trace$\,(A)$ is zero.

- Coordinate functions are harmonic: $\Delta_\Sigma x_j = 0$.

Classify all critical points to variational problem?

No compactness: Dilated surface $\lambda\Sigma$ has $A_{\lambda\Sigma} = \lambda^{-1} A_\Sigma$ so is also minimal.

Our surfaces are always *embedded*, i.e., $1-1$ immersions.

Three classical examples:

- **Planes** ($A = 0$; unit normal n is constant).

- **Catenoid** $x_3 = \cosh^{-1}\left(\sqrt{x_1^2 + x_2^2}\right)$.

- **Helicoid** $x_3 = \tan^{-1}(x_2/x_1)$.

Catenoid.

Helicoid.

Classical subject. Recent advances and solutions of long-standing questions. Many contributors.

Three areas stand out:

(I) Structure of properly embedded surfaces.

(II) Properness and Calabi-Yau conjectures.

(III) Constructions of new embedded surfaces.

Focus on (I) and (II), emphasizing work with Toby Colding, but also touching on F. Martin, W. Meeks, J. Perez, A. Ros, and H. Rosenberg.

Brief word on (III): new constructions. Three methods:

PDE: ("Gluing") N. Kapouleas; R. Mazzeo-F. Pacard.

Alg'c: ("Weierstrass") D. Hoffman-M. Weber-M. Wolf; M. Traizet.

Var'l: D. Hoffman-B. White.

Structure of **prop. emb. min. surf's**. Three cases:

(1) **Disks**; CM1–CM4, Annals of Math, 2004.

(2) **Planar domains**; CM5, preprint 2005.

(3) **Fixed genus**; CM5, preprint 2005.

A key point: Our results are **effective** $\Rightarrow$ apply to surfaces with boundary.

(1) plays a key role in (2) and (3).

Structure of an emb. min. **disk** Σ

Roughly: Main results of CM1 and CM2 $\Rightarrow$ Σ is either a graph or part of a double-spiral staircase.

A little more precisely:

- $|A|$ small relative to the distance to $\partial\Sigma$ $\Rightarrow$ graph. (Like the plane.)

- Otherwise, Σ is contained in a double-spiral staircase. (Like a big piece of the helicoid.)

Compactness of a seq. Σ_i of compact emb. min. disks with $\Sigma_i \subset B_{R_i}$ and $\partial \Sigma_i \subset \partial B_{R_i}$.

Theorem 1 *(CM4) If $R_i \to \infty$ and $|A|^2 \to \infty$ somewhere, then:*
A subseq. $\to$ foliation by parallel planes off of a transverse curve $\mathcal{S}$ where $|A| \to \infty$.

Note: This $\Rightarrow$ at each point, $|A| \to \infty$ or $|A| \to 0$.

Compactness in the classical examples:

- Dilated **Cat**'s $\to 2\times$ the flat plane $\setminus\{0\}$.

  Singularity $\{0\}$ is removable.

- Dilated **Hel**'s $\to$ Foliation $\{x_3 = \text{Const.}\} \setminus x_3$-axis (also removable).

The singular set $\mathcal{S}$ is where $|A| \to \infty$

Namely, if $A_i = A_{\Sigma_i}$, then we can pass to a subseq. so that

- $x \in \mathcal{S} \Rightarrow \forall\, r > 0$

$$\lim_{i \to \infty} \sup_{B_r(x)} |A_i|^2 \to \infty \,.$$

- $x \notin \mathcal{S} \Rightarrow \exists\, r_x > 0$ so that

$$\sup_i \sup_{B_{r_x}(x)} |A_i|^2 < \infty \,.$$

Easy Conseq.: Arzela-Ascoli $\Rightarrow \Sigma_i \setminus \mathcal{S} \to$ minimal lamination $\mathcal{L}'$ of $\mathbf{R}^3 \setminus \mathcal{S}$.

Three key ingredients

1. Local structure of CM1 and CM2 $\Rightarrow$ "Helicoids" forming near each $x \in \mathcal{S}$.

2. 1-sided curvature estimate of CM4 (using CM1-CM3) $\Rightarrow$ extends tangentially.

3. "Properness" (C-M, IMRN '02) $\Rightarrow$ extends transversely.

Compactness th'm and the 3 key ingred's play important role in other results. Three examples:

- W. Meeks and H. Rosenberg's Uniqueness of the helicoid:
  The plane and helicoid are the only complete properly emb. min. disks in $\mathbf{R}^3$.

- Meeks' Regularity: The Lipschitz curve $\mathcal{S}$ is actually a straight line $\perp$ to the planes.

- CM's effective version of P. Collin's theorem: Complete properly emb. min. annuli have finite total curvature.
  (So R. Schoen $\Rightarrow$ catenoids.)

General planar domains (i.e., topological disks with subdisks removed).

We already have seen the catenoid.

The other classical examples come from the 2-parameter family of singly periodic min. surf's discovered by Riemann. Roughly, this looks like a periodic collection of planes connected by "necks".

The param's can be thought of as:
1. Size of the necks.
2. Angle between the necks.

Riemann 1.

The local structure for planar domains is more complicated, but as before $\Rightarrow$ compactness.

Assume: Σ_i compact emb. min. planar domains with $\Sigma_i \subset B_{R_i}$ and $\partial \Sigma_i \subset \partial B_{R_i}$.

Theorem 2 *(CM) If $R_i \to \infty$ and $|A|^2 \to \infty$ somewhere, then*
subseq. $\to$ collection $\mathcal{L}$ of parallel planes off of set $\mathcal{S}$ where $|A|^2 \to \infty$.

We do not know in general that $\mathcal{L}$ is a foliation or that $\mathcal{S}$ is a curve.

Compactness in the classical examples:

- Dilated **Riemann examples** $\rightarrow$ Foliation $\{x_3 = \text{Const.}\} \setminus$ a line ℓ.

  ℓ <u>not</u> $\perp$ to planes. It is removable.

- **Riemann examples** as angle $\rightarrow \pi/2$ goes $\rightarrow$ Foliation $\{x_3 = \text{Const.}\} \setminus$ 2 vertical lines.

Unlike the disk case, now $\mathcal{S}$ consists of 2 types of sing's:

- $\mathcal{S}_{neck}$ where the injectivity radius $i(x)$ goes $\to 0$.
- $\mathcal{S}_{ulsc}$ where $|A|^2 \to \infty$ but $i(x) > 0$.

The No-mixing Thm:

Theorem 3 *(CM5) Either $\mathcal{S}_{neck} = \emptyset$ or $\mathcal{S}_{ulsc} = \emptyset$.*

When $\mathcal{S}_{neck} = \emptyset$: The ULSC case.

Theorem 4 *(CM5) If $\mathcal{S}_{neck} = \emptyset$, then:*
1. $\mathcal{L}$ is a foliation.
2. $\mathcal{S} = 2$ lines $\perp$ to the planes.

Note: We show $\mathcal{S} = 2$ curves; then Meeks' regularity $\Rightarrow$ lines.

The structure of complete planar dom's.

Beautiful series of papers by W. Meeks, J. Perez, and A. Ros; mention 2 of their results:

- M-P-R I: A complete properly emb. min. pd has bounded curvature.

- M-P-R II: A complete properly emb. min. pd with ∞-many ends has exactly 2 limit ends.

Properness and the Calabi-Yau Conj's.

A map is *proper* if the inverse image of any compact set is compact.

An immersed submanifold is proper if the immersion is a proper map. A **non-proper** example:

The polar graph ($\rho = \pi + \tan^{-1}(t), \theta = t$) is an ∞ spiral in a compact annulus.

Properness was assumed in every result thus far. Is this necessary?

This is essentially the point of the CY Conj's.

In 1965, E. Calabi asked:
1. Can we have a complete min. surf. in a ball?
2. Is every complete min. surf. between 2 planes flat?

S.S. Chern (1966) also asked 1.

S.T. Yau has revisited these questions several times.

Compare 1. and 2. with the Liouville theorem since the coordinate functions are harmonic.

Historical highlights on CY Conj's

Negative results:

A. Jorge-Xavier '80: Non-flat min. disk between 2 planes.

B. Nadirashvili '96: min. disk in a ball.
Other topological types: Lopez-Martin-Morales.

C. Alarcon-Ferrer-Martin '05: "Bad guys" dense.

Positive results before 2004:

A. Hoffman-Meeks '90: Proper and between 2 planes $\Rightarrow$ flat.

$\Rightarrow$ All comes down to properness.

Theorem 5 *(CM '04) Embedded min. surface with finite topology* $\Rightarrow$ *proper.*

In fact, this was a corollary of a stronger effective result.

Namely, we proved a chord-arc estimate, giving a lower bound for extrinsic distance in terms of intrinsic distance.

Thus, as intrinsic distance $\to \infty$, so does extrinsic distance $\Rightarrow$ properness.