

# Embedded minimal surfaces

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**Abstract.** The study of embedded minimal surfaces in  $\mathbf{R}^3$  is a classical problem, dating to the mid 1700's, and many people have made key contributions. We will survey a few recent advances, focusing on joint work with Tobias H. Colding of MIT and Courant, and taking the opportunity to focus on results that have not been highlighted elsewhere.

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## 1. Introduction

An immersed surface  $\Sigma$  in  $\mathbf{R}^3$  is said to be *minimal* if it has zero mean curvature and is *embedded* if the immersion is injective. The study of embedded minimal surfaces in  $\mathbf{R}^3$  is a classical problem, dating to the mid 1700's, and many people have made key contributions.

Many of the recent results have been surveyed elsewhere and we will take the opportunity to highlight results that have not been as well covered, concentrating on recent joint work with Tobias H. Colding of MIT and the Courant Institute. We will also briefly cover recent important results of W. Meeks and H. Rosenberg and of W. Meeks, J. Perez, and A. Ros. We refer to the following surveys for other perspectives:

- For more on the structure of properly embedded minimal surfaces, see the joint expository article [11] with Tobias H. Colding as well as the surveys [27] of W. Meeks and J. Perez, [35] of J. Perez, [36] of H. Rosenberg, as well as the joint surveys [14], [16], and [17] with Tobias H. Colding.
- For the construction of embedded minimal surfaces, see the surveys [22] of D. Hoffman, M. Weber, and M. Wolf, [23] of N. Kapouleas, and [38] of M. Traizet.
- For properness of minimal surfaces and the Calabi-Yau Conjectures, see the paper [15] as well as the surveys [24] of F. Martin, [16], [17], and [35].

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**1.1. Embedded minimal surfaces of fixed genus.** We have chosen to concentrate on the following central question:

- Can one compactify the space of embedded minimal surfaces of fixed genus?

Roughly speaking, we show in [8] that a sequence of embedded minimal surfaces with fixed genus has a subsequence that converges away from a singular set to a collection of parallel planes. The precise structure of the singular set and of the surfaces near the singular set depends on the topology of the surfaces. Consequently, we consider three separate cases:

1. When the surfaces are disks.
2. When the surfaces are (non-simply connected) planar domains; i.e., the case of genus zero.
3. When the surfaces have a fixed non-zero genus.

The case of disks was completed in [4], [5], [6] and [7] and plays a key role in the other two cases as well; the case of disks was surveyed in [14] and [16]. The other two cases, which were completed in [8], will be one of the focal points of this survey.

A key step in the compactness results for embedded minimal surfaces of fixed genus is a structure result that describes what these surfaces look like. We have chosen to focus on the compactness theorems rather than the underlying structure results, largely because it serves as a unifying theme and allows us to simplify some of the statements. Roughly speaking, two main structure theorems for (non-simply connected) embedded minimal planar domains from [8] are:

- Any such surface without small necks can be obtained by gluing together two oppositely-oriented double spiral staircases.
- Any such surface with small necks can be decomposed into “pairs of pants” by cutting the surface along a collection of short curves. After the cutting, we are left with graphical pieces that are defined over a disk with either one or two sub-disks removed (a topological disk with two sub-disks removed is called a *pair of pants*).

Both of these structures occur as different extremes in the two-parameter family of minimal surfaces known as the Riemann examples.

**1.2. Embedded minimal annuli.** The simplest example of a non-simply connected planar domain is of course an annulus. In [10], we obtained a precise description of what an embedded minimal annulus in a ball must look like - roughly speaking, it must look like catenoid. This illustrates a few of the ideas for the general pair of pants decomposition of [8] in a relatively simple setting. This description can be thought of as an effective version of the main theorems of [18] and [13]; i.e., [10] applies to an annulus  $\Sigma$  with  $\partial\Sigma \subset \partial B_r(0)$  and as  $r$  goes to infinity we recover the results of [18] and [13].

**1.3. Properness and removable singularities.** The next result that we will highlight is the proof of “properness” in [9]. This properness was used in [7] to analyze a neighborhood of each singular point, showing that an entire neighborhood is foliated by limit planes. This can be viewed as a removable singularity theorem for minimal laminations. The proof of properness in [9] works only in the global case where we have a sequence of embedded minimal disks in a sequence of expanding balls whose radii tend to infinity - the local case is where the disks are in a fixed ball. Perhaps surprisingly, it turned out that properness can fail in the local case: In the local case, we can get limits with non-removable singularities. One local example with non-removable singularities is constructed in [12].

**1.4. The global structure of complete embedded minimal surfaces in  $\mathbf{R}^3$ .** As mentioned above, there have been many important recent developments in the field. We will survey two of these where the results of [4]–[8] play a role:

- The uniqueness of the helicoid, proven by W. Meeks and H. Rosenberg in [31].
- The curvature bound for embedded minimal planar domains with bounded horizontal flux proven by W. Meeks, J. Perez, and A. Ros in [28].

The uniqueness of the helicoid solved a long-standing problem that was largely considered unapproachable until recently and also has many applications. We will sketch the proof and explain how the lamination theorem and one-sided curvature estimate played a key role.

The curvature bound of [28] was the key step in solving an old conjecture of J. Nitsche and an important step for understanding the moduli space of embedded minimal planar domains. We will explain the result, give an idea why it should be true, and explain how the compactness theorems of [8] play a role in the proof.

We should point out that there is a key distinction between these two results and the other results that we have discussed: these results both use in an essential way that the surfaces are complete and without boundary.

## 2. Minimal surfaces

An immersed surface  $\Sigma \subset \mathbf{R}^3$  is *minimal* if it is a critical point for area, i.e., if it has zero mean curvature. The *mean curvature* is the trace of the second fundamental form  $A$ ; recall that the eigenvalues of  $A$  are called the *principal curvatures*. Our surface  $\Sigma$  will always be embedded and will have a well-defined unit normal  $\mathbf{n}$ . The map

$$\mathbf{n} : \Sigma \rightarrow \mathbf{S}^2 \tag{1}$$

is called the *Gauss map*. Note that  $A$  is the differential of the Gauss map.

Observe that if  $\Sigma \subset \mathbf{R}^3$  is minimal, then so is every rigid motion of  $\Sigma$ . Furthermore, so is a dilation of  $\Sigma$ , i.e., so is the surface

$$\lambda \Sigma = \{\lambda x \mid x \in \Sigma\}. \quad (2)$$

This is because dilating  $\Sigma$  by  $\lambda$  dilates the second fundamental form by  $\lambda^{-1}$ .

Note that minimal surfaces are not necessarily area-minimizing. A surface is *stable* if it satisfies the second derivative test; obviously, area-minimizing surfaces are stable.

**2.1. Classical minimal surfaces.** The simplest example of a minimal surface is a flat plane (where the unit normal is constant and, hence, where  $A = 0$ ).

The only non-trivial rotationally invariant minimal surface is the *catenoid* (discovered in 1776), i.e., the minimal surface in  $\mathbf{R}^3$  parametrized by

$$(\cosh s \cos t, \cosh s \sin t, s) \text{ where } s, t \in \mathbf{R}. \quad (3)$$

More precisely, since dilations preserve minimality, there is a one-parameter family of catenoids (modulo rigid motions) given by

$$\lambda (\cosh s \cos t, \cosh s \sin t, s) \text{ where } s, t \in \mathbf{R}. \quad (4)$$

The *helicoid* (also discovered in 1776) is the minimal surface  $\Sigma$  in  $\mathbf{R}^3$  parametrized by

$$(s \cos t, s \sin t, t) \text{ where } s, t \in \mathbf{R}. \quad (5)$$

Note that the helicoid is a “double-spiral staircase”, consisting of a straight line in each horizontal plane where these lines rotate at constant speed. It can also be thought of as the union of the “graphs” of the functions  $\theta$  and  $\theta + \pi$  together with the vertical axis. We will make this last characterization more precise later when we introduce the notion of a multi-valued graph.

The catenoid can be thought of as “two planes glued together along a small neck.” Surprisingly, by a theorem of F. Lopez and A. Ros, it is impossible to glue together any other *finite* number of planes to get a complete properly embedded minimal planar domain. However, the *Riemann examples* (constructed by Riemann around 1860) give a periodic collection of horizontal planes glued together along small necks. This is actually (modulo rigid motions) a two parameter family of surfaces, where the parameters can roughly be thought of as

- the size of the necks (or injectivity radius), and
- the angle from one to the next.

As the angle goes to zero, the necks get further and further apart and the family degenerates to a collection of catenoids. As the angle goes to  $\pi/2$ , the necks become virtually on top of each other and the family degenerates to the union of two oppositely oriented helicoids. There are very pretty pictures of this available from David Hoffman’s web page:

<http://www.msri.org/about/sgp/jim/geom/minimal/library/riemann/index.html>

### 3. Embedded minimal surfaces with fixed genus

As mentioned, we will focus on compactness theorems for a sequence  $\Sigma_i \subset \mathbf{R}^3$  of embedded minimal surfaces. There are various notions of weak convergence (e.g., as currents or varifolds). However, for us, the sequence  $\Sigma_i$  converges to a surface  $\Sigma_\infty$  at a point  $x \in \Sigma$  if there is a ball  $B_r(x)$  so that:

- For every  $i$  sufficiently large,  $B_r(x) \cap \Sigma_i$  is a (connected) graph over a (subset of) the tangent plane  $T_x \Sigma_\infty$  of a function  $u_i$ .
- As  $i \rightarrow \infty$ , the functions  $u_i$  converge smoothly to a function  $u_\infty$  where  $B_r(x) \cap \Sigma_\infty$  is the graph of  $u_\infty$ .

Notice that there are two obvious necessary conditions for the sequence  $\Sigma_i$  to converge in this sense: The curvatures and areas of the sequence must be locally bounded.

It is not hard to see that the lack of a local area bound is not such a serious problem as long as we have embeddedness. Namely, if we have a uniform curvature bound near  $x$ , then the components of  $B_r(x) \cap \Sigma_i$  are well-approximated by their tangent planes for  $r$  small. Embeddedness then implies that all of these tangent planes must be almost parallel. In particular, these components are all graphs over the same plane of functions with a uniform  $C^1$  bound. We can use the Arzela-Ascoli theorem to pass to a subsequence that “converges” to a collection of minimal surfaces that do not cross. The strong maximum principle then implies that two of these limit surfaces must be identical if they touch at all, i.e., they are like the leaves of a foliation. This sort of structure is called a *lamination*.

The failure of the curvature bound is a more serious problem and will force us to allow for a singular set where the sequence simply does not converge smoothly. The simplest example of this is a sequence of rescalings  $\lambda_i \Sigma$  with  $\lambda_i \rightarrow 0$  of a fixed non-flat complete embedded minimal surface  $\Sigma$ . This scales the curvature by the factor  $\lambda_i^{-1}$  and, thus, will force the curvature to blow up at the origin. For example, a sequence of rescaled catenoids converges with multiplicity two to the punctured plane. The convergence is smooth except at 0 where  $|A|^2 \rightarrow \infty$ . Notice that 0 is a removable singularity for the limit.

It follows from Choi and Schoen, [1], that a similar singular compactness result holds as long as we assume a uniform bound on the total curvature:

A subsequence converges smoothly with finite multiplicity away from a finite set of singular points; these singular points are then removable singularities for the limit surface.<sup>1</sup>

The situation is more complicated when there is no a priori total curvature bound. For example, if we take a sequence of rescaled helicoids, then the curvature blows up along the entire vertical axis but is bounded away from this axis. Thus, we get that

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<sup>1</sup>In fact, one can say a good deal more about the convergence and the structure of the limit; see the 1995 paper of A. Ros in Indiana Math.

- The intersection of the rescaled helicoids with a ball away from the vertical axis gives a collection of graphs over the plane  $\{x_3 = 0\}$ . As  $i \rightarrow \infty$ , these graphs become flat and horizontal.
- The intersection of the rescaled helicoids with a ball centered on the vertical axis gives a double spiral staircase, rotating faster and faster as  $i \rightarrow \infty$ .

In particular, the sequence of rescaled helicoids converges away from the vertical axis to a foliation by flat parallel planes.

**Remark 3.1.** The same thing happens when one rescales any surface asymptotic to the helicoid - such as the genus one helicoid constructed by D. Hoffman, M. Weber, and M. Wolf in [21].

If we do the same rescaling to a fixed surface in the family of Riemann examples, then we get convergence away from a line to a foliation by horizontal planes. In this case, the line is not perpendicular to the planes.

However, unlike the catenoid and helicoid, the Riemann examples are a two-parameter family. By choosing the two parameters appropriately, one can produce sequences of Riemann examples that illustrate both of the two structure theorems:

1. If we take a sequence of Riemann examples where the neck size is fixed and the angles go to  $\frac{\pi}{2}$ , then the surfaces with angle near  $\frac{\pi}{2}$  can be obtained by gluing together two oppositely-oriented double spiral staircases. Each double spiral staircase looks like a helicoid. This sequence of Riemann examples converges to a foliation by parallel planes. The convergence is smooth away from the axes of the two helicoids (these two axes are the singular set where the curvature blows up).
2. Suppose now that we take a sequence of examples where the neck sizes go to zero. In this case, the surfaces can be cut along short curves into collections of graphical pairs of pants. The short curves converge to singular points where the curvature blows up and the graphical pieces converge to flat planes except at these points.

## 4. [8]: Compactness of embedded minimal surfaces with fixed genus

We turn next to the main compactness results of [8] for embedded minimal surfaces with fixed genus. We will restrict our discussion to the case of planar domains, i.e., when the surfaces have genus zero, to simplify things. In any case, the general case of fixed genus requires only minor changes.

**In this section,  $\Sigma_i \subset B_{R_i} \subset \mathbf{R}^3$  is a sequence of compact embedded minimal planar domains with  $\partial\Sigma_i \subset \partial B_{R_i}$ . Moreover, we will assume that  $R_i \rightarrow \infty$ .**

The singular set  $\mathcal{S}$  is defined to be the set of points where the curvature is blowing up. That is, a point  $y$  in  $\mathbf{R}^3$  is in  $\mathcal{S}$  for a sequence  $\Sigma_i$  if

$$\sup_{B_r(y) \cap \Sigma_i} |A|^2 \rightarrow \infty \text{ as } i \rightarrow \infty \text{ for all } r > 0. \quad (6)$$

It is not hard to see that we can pass to a subsequence so that  $\mathcal{S}$  is well-defined and, furthermore, if  $x \notin \mathcal{S}$ , then there exists  $r_x > 0$  so that

$$\sup_i \sup_{B_{r_x}(x) \cap \Sigma_i} |A| < \infty. \quad (7)$$

**4.1. The finer structure of  $\mathcal{S}$ : Where the topology concentrates.** Sequences of planar domains which are not simply connected are, after passing to a subsequence, naturally divided into two separate cases depending on whether or not the topology is concentrating at points. To distinguish between these cases, we will say that a sequence of surfaces  $\Sigma_i^2 \subset \mathbf{R}^3$  is *uniformly locally simply connected* (or ULSC) if for each  $x \in \mathbf{R}^3$ , there exists a constant  $r_0 > 0$  (depending on  $x$ ) so that for every surface  $\Sigma_i$

$$\text{each connected component of } B_{r_0}(x) \cap \Sigma_i \text{ is a disk.} \quad (8)$$

For instance, a sequence of rescaled catenoids where the necks shrink to zero is not ULSC, whereas a sequence of rescaled helicoids is.

Another way of locally distinguishing sequences where the topology does not concentrate from sequences where it does comes from analyzing the singular set. The singular set  $\mathcal{S}$  consists of two types of points. The first type is roughly modelled on rescaled helicoids and the second on rescaled catenoids:

- A point  $y$  in  $\mathbf{R}^3$  is in  $\mathcal{S}_{ulsc}$  if the curvature for the sequence  $\Sigma_i$  blows up at  $y$  and the sequence is ULSC in a neighborhood of  $y$ .
- A point  $y$  in  $\mathbf{R}^3$  is in  $\mathcal{S}_{neck}$  if the sequence is not ULSC in any neighborhood of  $y$ . In this case, a sequence of closed non-contractible curves  $\gamma_i \subset \Sigma_i$  converges to  $y$ .

The sets  $\mathcal{S}_{neck}$  and  $\mathcal{S}_{ulsc}$  are obviously disjoint and the curvature blows up at both, so  $\mathcal{S}_{neck} \cup \mathcal{S}_{ulsc} \subset \mathcal{S}$ . An easy argument (proposition I.0.19 in [6]) shows that, after passing to a further subsequence, we can assume that

$$\mathcal{S} = \mathcal{S}_{neck} \cup \mathcal{S}_{ulsc}. \quad (9)$$

Note that  $\mathcal{S}_{neck} = \emptyset$  is equivalent to that the sequence is ULSC as is the case for sequences of rescaled helicoids. On the other hand,  $\mathcal{S}_{ulsc} = \emptyset$  for sequences of rescaled catenoids. (These definitions of  $\mathcal{S}_{ulsc}$  and  $\mathcal{S}_{neck}$  are specific to the genus zero case that we are focusing on now; the definitions in the fixed genus case can be found in section 1.1 of [8].)

**4.2. Compactness away from  $\mathcal{S}$ .** If we combine the local curvature bound (7) away from  $\mathcal{S}$  and a variation on the Arzela-Ascoli theorem, we can pass to a subsequence so that the  $\Sigma_i$ 's converge away from  $\mathcal{S}$  to a limit lamination  $\mathcal{L}'$  of  $\mathbf{R}^3 \setminus \mathcal{S}$ .

The leaves of  $\mathcal{L}'$  are smooth, but not necessarily complete, surfaces. To make this precise, we define the closure  $\Gamma_{Clos}$  of a leaf  $\Gamma$  of  $\mathcal{L}'$  to be the union of the closures of all bounded (intrinsic) geodesic balls in  $\Gamma$ ; that is, we fix a point  $x_\Gamma \in \Gamma$  and set

$$\Gamma_{Clos} = \bigcup_r \overline{\mathcal{B}_r(x_\Gamma)}, \quad (10)$$

where  $\overline{\mathcal{B}_r(x_\Gamma)}$  is the closure of  $\mathcal{B}_r(x_\Gamma)$  as a subset of  $\mathbf{R}^3$ .

Clearly, a leaf  $\Gamma$  is complete if and only if  $\Gamma_{Clos} = \Gamma$  and we always have that

$$\Gamma_{Clos} \setminus \Gamma \subset \mathcal{S}. \quad (11)$$

The incomplete leaves of  $\Gamma$  can be divided into several types, depending on how  $\Gamma_{Clos}$  intersects  $\mathcal{S}$ :

- Collapsed leaves where  $\Gamma_{Clos} \cap \mathcal{S}_{ulsc}$  contains a removable singularity for  $\Gamma$ .
- Leaves  $\Gamma$  with  $\Gamma_{Clos} \cap \mathcal{S}_{ulsc} \neq \emptyset$ , but where  $\Gamma$  does not have a removable singularity. This would occur, for example, if  $\Gamma$  spirals infinitely into the collapsed leaf through  $\Gamma_{Clos} \cap \mathcal{S}_{ulsc}$ . (We show in [8] that this does not occur.)
- Leaves  $\Gamma$  where  $\Gamma_{Clos} \setminus \Gamma \subset \mathcal{S}_{neck}$ ; these obviously don't occur in the ULSC case.

**4.3. Disks.** Before discussing the general ULSC case, it is useful to recall the case of disks. One consequence of [4]–[7] is that there are only two local models for ULSC sequences of embedded minimal surfaces. That is, locally in a ball in  $\mathbf{R}^3$ , one of following holds:

- The curvatures are bounded and the surfaces are locally graphs over a plane.
- The curvatures blow up and the surfaces are locally double spiral staircases.

Both of these cases are illustrated by taking a sequence of rescalings of the helicoid; the first case occurs away from the axis, while the second case occurs on the axis.

Using in part this local description, we were able to prove that any sequence of embedded minimal disks with curvatures blowing up has a subsequence that converges to a foliation by parallel planes. This convergence is away from a Lipschitz curve  $\mathcal{S}$  that is transverse to the planes. (See the appendix for the precise statements.)

**4.4. Planar domains: the general structure theorems.** We will show that every sequence  $\Sigma_i$  has a subsequence that is either ULSC or for which  $\mathcal{S}_{ulsc}$  is empty. This is the next “no mixing” theorem. We will see later that these two different cases give two very different structures.

**Theorem 4.1** (No mixing theorem, [8]). *If the  $\Sigma_i$ 's are genus zero, then there is a subsequence with either  $\mathcal{S}_{ulsc} = \emptyset$  or  $\mathcal{S}_{neck} = \emptyset$ .*

Common for both the ULSC case and the case where  $\mathcal{S}_{ulsc}$  is empty is that the limits are always laminations by flat parallel planes and the singular sets are always closed subsets contained in the union of the planes. This is the content of the next theorem:

**Theorem 4.2** (Planar lamination theorem, [8]). *If the  $\Sigma_i$ 's are genus zero and*

$$\sup_{B_1 \cap \Sigma_i} |A|^2 \rightarrow \infty, \quad (12)$$

*then there exists a subsequence  $\Sigma_j$ , a lamination  $\mathcal{L} = \{x_3 = t\}_{\{t \in \mathcal{I}\}}$  of  $\mathbf{R}^3$  by parallel planes (where  $\mathcal{I} \subset \mathbf{R}$  is a closed set), and a closed nonempty set  $\mathcal{S}$  in the union of the leaves of  $\mathcal{L}$  such that after a rotation of  $\mathbf{R}^3$ :*

- (A) *For each  $1 > \alpha > 0$ ,  $\Sigma_j \setminus \mathcal{S}$  converges in the  $C^\alpha$ -topology to the lamination  $\mathcal{L} \setminus \mathcal{S}$ .*
- (B)  *$\sup_{B_r(x) \cap \Sigma_j} |A|^2 \rightarrow \infty$  as  $j \rightarrow \infty$  for all  $r > 0$  and  $x \in \mathcal{S}$ . (The curvatures blow up along  $\mathcal{S}$ .)*

**4.5. Planar domains: the fine structure theorems.** We will assume here that the  $\Sigma_i$ 's are not disks (recall that the case of disks was dealt with in [4]–[7]). In particular, we will assume that for each  $i$ , there exists some  $y_i \in \mathbf{R}^3$  and  $s_i > 0$  so that

$$\text{some component of } B_{s_i}(y_i) \cap \Sigma_i \text{ is not a disk.} \quad (13)$$

Moreover, if the non-simply connected balls  $B_{s_i}(y_i)$  “run off to infinity” (i.e., if each connected component of  $B_{R'_i}(0) \cap \Sigma_i$  is a disk for some  $R'_i \rightarrow \infty$ ), then the results of [4]–[7] apply. Therefore, after passing to a subsequence, we can assume that the surfaces are uniformly not disks, namely, that there exists some  $R > 0$  so that (13) holds with  $s_i = R$  and  $y_i = 0$  for all  $i$ .

In view of Theorem 4.1 and the earlier results for disks, it is natural to first analyze sequences that are ULSC, so where  $\mathcal{S}_{neck} = \emptyset$ , and second analyze sequences where  $\mathcal{S}_{ulsc}$  is empty. We will do this next.

**4.6. ULSC sequences.** Loosely speaking, our next result shows that when the sequence is ULSC (but not simply connected), a subsequence converges to a foliation by parallel planes away from two lines  $\mathcal{S}_1$  and  $\mathcal{S}_2$ . The lines  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are disjoint and orthogonal to the leaves of the foliation and the two lines are precisely the points where the curvature is blowing up. This is similar to the case of disks, except that we get two singular curves for non-disks as opposed to just one singular curve for disks.

**Theorem 4.3** (ULSC compactness, [8]). *Let a sequence  $\Sigma_i$ , limit lamination  $\mathcal{L}$ , and singular set  $\mathcal{S}$  be as in Theorem 4.2. Suppose that each  $\Sigma_i$  satisfies (13) with*

$s_i = R > 1$  and  $y_i = 0$ . If every  $\Sigma_i$  is ULSC and

$$\sup_{B_1 \cap \Sigma_i} |A|^2 \rightarrow \infty, \quad (14)$$

then the limit lamination  $\mathcal{L}$  is the foliation  $\mathcal{F} = \{x_3 = t\}_t$  and the singular set  $\mathcal{S}$  is the union of two disjoint lines  $\mathcal{S}_1$  and  $\mathcal{S}_2$  such that:

( $C_{ulsc}$ ) Away from  $\mathcal{S}_1 \cup \mathcal{S}_2$ , each  $\Sigma_j$  consists of exactly two multi-valued graphs spiraling together. Near  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , the pair of multi-valued graphs form double spiral staircases with opposite orientations at  $\mathcal{S}_1$  and  $\mathcal{S}_2$ . Thus, circling only  $\mathcal{S}_1$  or only  $\mathcal{S}_2$  results in going either up or down, while a path circling both  $\mathcal{S}_1$  and  $\mathcal{S}_2$  closes up.

( $D_{ulsc}$ )  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are orthogonal to the leaves of the foliation.

**Remark 4.4.** See Appendix A for the definition of a multi-valued graph. Roughly speaking a multi-valued graph is locally a graph over a subset of a plane, but fails to be a global graph since the projection to the plane is not one-to-one.

**4.7. Sequences that are not ULSC.** When the sequence is no longer ULSC, one can get other types of curvature blow-up by considering the family of embedded minimal planar domains known as the Riemann examples. Recall that, modulo translations and rotations, this is a two-parameter family of periodic minimal surfaces, where the parameters can be thought of as the size of the necks and the angle from one fundamental domain to the next.

With these examples in mind, we are now ready to state our second main structure theorem describing the case where  $\mathcal{S}_{ulsc}$  is empty.

**Theorem 4.5** ([8]). *Let a sequence  $\Sigma_i$ , limit lamination  $\mathcal{L}$ , and singular set  $\mathcal{S}$  be as in Theorem 4.2. If  $\mathcal{S}_{ulsc} = \emptyset$  and*

$$\sup_{B_1 \cap \Sigma_i} |A|^2 \rightarrow \infty, \quad (15)$$

then  $\mathcal{S} = \mathcal{S}_{neck}$  by (9) and

( $C_{neck}$ ) Each point  $y$  in  $\mathcal{S}$  comes with a sequence of graphs in  $\Sigma_j$  that converge to the plane  $\{x_3 = x_3(y)\}$ . The convergence is in the  $C^\infty$  topology away from the point  $y$  and possibly also one other point in  $\{x_3 = x_3(y)\} \cap \mathcal{S}$ . If the convergence is away from one point, then these graphs are defined over annuli; if the convergence is away from two points, then the graphs are defined over disks with two subdisks removed.

**4.8. An overview of the proofs: The ULSC case.** A key point will be that the results of [4]–[7] for disks will give a sequence of multi-valued graphs in the  $\Sigma_j$ 's near each point  $x \in \mathcal{S}_{ulsc}$ . Moreover, these multi-valued graphs close up in the limit to give a leaf of  $\mathcal{L}'$  which extends smoothly across  $x$ . Such a leaf is said to be *collapsed*; in a neighborhood of  $x$ , the leaf can be thought of as a limit

of double-valued graphs where the upper sheet collapses onto the lower. We show that every collapsed leaf is stable, has at most two points of  $\mathcal{S}_{ulsc}$  in its closure, and these points are removable singularities. These results on collapsed leaves are applied first in the USLC case and then again to get the structure of the ULSC regions of the limit in general, i.e., (C2) and (D) in Theorem C.1.

Roughly speaking, there are two main steps to the proof of Theorem 4.3:

1. Show that each collapsed leaf is in fact a plane punctured at two points of  $\mathcal{S}$  and, moreover, the sequence has the structure of a double spiral staircase near both of these points, with opposite orientations at the two points.
2. Show that leaves which are nearby a collapsed leaf of  $\mathcal{L}'$  are also planes punctured at two points of  $\mathcal{S}$ . (We call this “properness”.)

**4.9. An overview of the proofs: The general structure.** Theorem 4.5, as well as Theorem 4.1, are proven by first analyzing sequences of minimal surfaces without any assumptions on the sets  $\mathcal{S}_{ulsc}$  and  $\mathcal{S}_{neck}$ . The precise statement of this general theorem is given in Appendix C. We will give an overview of the theorem next.

In this general case, we show that a subsequence converges to a lamination  $\mathcal{L}'$  divided into regions where Theorem 4.3 holds and regions where Theorem 4.5 holds. This convergence is in  $C^{1,1}$  topology away from the singular set  $\mathcal{S}$  where the curvature blows up. Moreover, each point of  $\mathcal{S}$  comes with a plane and these planes are essentially contained in  $\mathcal{L}'$ . The set of heights of the planes is a closed subset  $\mathcal{I} \subset \mathbf{R}$  but may not be all of  $\mathbf{R}$  as it was in Theorem 4.3 and may not even be connected. The behavior of the sequence is different at the two types of singular points in  $\mathcal{S}$  - the set  $\mathcal{S}_{neck}$  of “catenoid points” and the set  $\mathcal{S}_{ulsc}$  of ULSC singular points. We will see that  $\mathcal{S}_{ulsc}$  consists of a union of Lipschitz curves transverse to the lamination  $\mathcal{L}$ . This structure of  $\mathcal{S}_{ulsc}$  implies that the set of heights in  $\mathcal{I}$  which intersect  $\mathcal{S}_{ulsc}$  is a union of intervals; thus this part of the lamination is foliated. In contrast, we will not get any structure of the set of “catenoid points”  $\mathcal{S}_{neck}$ . Given a point  $y$  in  $\mathcal{S}_{neck}$ , we will get a sequence of graphs in  $\Sigma_j$  converging to a plane through  $y$ . This convergence will be in the smooth topology away from either one or two singular points, one of which is  $y$ . Moreover, this limit plane through  $y$  will be a leaf of the lamination  $\mathcal{L}$ .

The key steps for proving the general structure theorem are the following:

1. Finding a stable plane through each point of  $\mathcal{S}_{neck}$ . This plane will be a limit of a sequence of stable graphical annuli that lie in the complement of the surfaces.
2. Finding graphs in  $\Sigma_j$  that converge to a plane through each point of  $\mathcal{S}_{neck}$ . To do this, we look in regions between consecutive necks and show that in any such region the surfaces are ULSC. The one-sided curvature estimate will then allow us to show that these regions are graphical.
3. Using (1) and (2) we then analyze the ULSC regions of a limit. That is, we show that if the closure of a leaf in  $\mathcal{L}'$  intersects  $\mathcal{S}_{ulsc}$ , then it has a

neighborhood that is ULSC. This will allow us to use the argument for the proof of Theorem 4.3 to get the same structure for such a neighborhood as we did in case where the entire surfaces were ULSC.

The main point left in Theorem 4.5, which is not included in this general compactness theorem, is to prove that every leaf of the lamination  $\mathcal{L}$  in Theorem 4.5 is a plane. In contrast, the general compactness theorem gives a plane through each point of  $\mathcal{S}_{neck}$ , but does not claim that the leaves of  $\mathcal{L}'$  are planar.

Finally, since the no mixing theorem implies that Theorem 4.3 and Theorem 4.5 cover all cases, Theorem 4.2 will be a corollary of these two theorems.

## 5. The structure of embedded minimal annuli

We turn next to a local structure theorem for embedded minimal annuli that, roughly speaking, shows that they must look like catenoids. Namely, the main theorem of [10] proves that any embedded minimal annulus in a ball (with boundary in the boundary of the ball and) with a small neck can be decomposed by a simple closed geodesic into two graphical sub-annuli. Moreover, there is a sharp bound for the length of this closed geodesic in terms of the separation (or height) between the graphical sub-annuli. This serves to illustrate the “pair of pants” decomposition from [8] in the special case where the embedded minimal planar domain is an annulus.

The precise statement of this decomposition for annuli is:

**Theorem 5.1** (Main Theorem, [10]). *There exist  $\epsilon > 0$ ,  $C_1, C_2, C_3 > 1$  so: If  $\Sigma \subset B_R \subset \mathbf{R}^3$  is an embedded minimal annulus with  $\partial\Sigma \subset \partial B_R$  and  $\pi_1(B_{\epsilon R} \cap \Sigma) \neq 0$ , then there is a simple closed geodesic  $\gamma \subset \Sigma$  of length  $\ell$  so that:*

- *The curve  $\gamma$  splits the connected component of  $B_{R/C_1} \cap \Sigma$  containing it into annuli  $\Sigma^+$  and  $\Sigma^-$ , each with  $\int |A|^2 \leq 5\pi$ .*
- *Each of  $\Sigma^\pm \setminus \mathcal{T}_{C_2\ell}(\gamma)$  is a graph with gradient  $\leq 1$ .*
- *$\ell \log(R/\ell) \leq C_3 h$  where the separation  $h$  is given by*

$$h = \min_{x_\pm \in \partial B_{R/C_1} \cap \Sigma^\pm} |x_+ - x_-|. \quad (16)$$

Here  $\mathcal{T}_s(S) \subset \Sigma$  denotes the intrinsic  $s$ -tubular neighborhood of a subset  $S \subset \Sigma$ .

**5.1. A sketch of the proof.** We will next give a brief sketch of the proof of the decomposition theorem, Theorem 5.1. The starting point is to use the hypothesis  $\pi_1(B_{\epsilon R} \cap \Sigma) \neq 0$  and a barrier argument to find a stable graph  $\Gamma_0$  that is defined over an annulus and disjoint from  $\Sigma$ . The stable graph  $\Gamma_0$  will allow us to divide  $\Sigma$  into two pieces, one on each side of  $\Gamma_0$ . To do this, we first fix a simple closed  $\tilde{\gamma} \subset B_{\epsilon R} \cap \Sigma$  that separates the two boundary components of  $\Sigma$ . The curve

$\tilde{\gamma}$  is contained in a small extrinsic ball, but there is no a priori reason why it must be short.<sup>2</sup> A barrier argument using a result of Meeks and Yau then gives a stable embedded minimal annulus  $\Gamma$  that separates the two boundary components of  $\Sigma$  and where  $\tilde{\gamma}$  is one component of the boundary  $\partial\Gamma$  and the other component is in  $\partial B_R$ . Finally, Theorem 0.3 of [6] then implies that  $\Gamma$  contains the desired graph  $\Gamma_0$ ; this should be compared with the well-known result of D. Fischer-Colbrie in the complete case.

We will see next that each half of  $\Sigma$ , i.e., the part above  $\Gamma_0$  and the part below  $\Gamma_0$ , is itself a graph away from the boundary of  $\Gamma_0$ . This part of the argument applies more generally to an “annular end” of a minimal surface. We will prove that each half of  $\Sigma$  contains a graph by showing that it must contain large locally graphical pieces and then using embeddedness to see that these pieces must be global graphs (i.e., the projection down is one-to-one). This follows by combining three facts:

- (1) The one-sided curvature estimate of [4]–[7] gives a scale-invariant curvature estimate for  $\Sigma$ 's in a narrow cone about the graph  $\Gamma_0$ . This requires that we know that each component of  $\Sigma$  in balls away from the origin is a disk; this can be seen from the maximum principle.
- (2) Using (1), the gradient estimate gives a narrower cone about  $\Gamma_0$  where  $\Sigma$  is locally graphical. This is because (1) implies that the surface is well-approximated by its tangent plane and, since it cannot cross  $\Gamma_0$ , it must be almost parallel to  $\Gamma_0$ .
- (3) As long as  $\epsilon$  is small enough, each half of  $\Sigma$  must intersect any narrow cone about  $\Gamma_0$ . This was actually proven in lemma 3.3 of [9] that gave the existence of low points in a connected minimal surface contained on one side of a plane and with interior boundary close to this plane.<sup>3</sup>

Step (3) allows us to find very flat regions in  $\Sigma$  near  $\Gamma_0$ , we can then repeatedly apply the Harnack inequality to build this out into large locally graphical regions that stay inside the narrow cone about  $\Gamma_0$ . These locally graphical regions piece together to give a graph over an annulus; the other possibility would be to form a multi-valued graph, but this is impossible since such a multi-valued graph would be forced to spiral infinitely (since it cannot cross itself and also cannot cross the stable graph  $\Gamma_0$ ).

Finally, the last step of the proof is to use a blow up argument to get the precise bounds on the length of the curve  $\gamma$ .

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<sup>2</sup>However, the chord arc bounds in the later paper [15] could now be used to bound the length.

<sup>3</sup>The argument for this was by contradiction. Namely, if there were no low points, then we would get a contradiction from the strong maximum principle by first sliding a catenoid up under the surface and then sliding the catenoid horizontally away, eventually separating two boundary components of the surface. Here the strong maximum principle is used to keep the sliding catenoids and the surface disjoint. See, for instance, corollary 1.18 in [2] for a precise statement of the strong maximum principle.

**5.2. Complete properly embedded minimal annuli.** The decomposition of properly embedded minimal annuli given by Theorem 5.1 can be viewed as a local version of well-known global results of P. Collin, [18], and Colding and the author, [13], on annular ends.

To explain these global results, recall that  $\Sigma$  is said to have *finite topology* if it is homeomorphic to a closed Riemann surface with a finite number of punctures; the genus of  $\Sigma$  is then the genus of this Riemann surface and the number of punctures is the number of ends. It follows that a neighborhood of each puncture corresponds to a properly embedded annular end of  $\Sigma$ . Perhaps surprisingly at first, the more restrictive case is when  $\Sigma$  has more than one end. The reason for this is that a barrier argument gives a stable minimal surface between any pair of ends. Such a stable surface is then asymptotic to a plane (or catenoid), essentially forcing each end to live in a half-space. Using this restriction, P. Collin proved:

**Theorem 5.2** (Main theorem, [18]). *Each end of a complete properly embedded minimal surface with finite topology and at least two ends is asymptotic to a plane or catenoid.*

In particular, such a  $\Sigma$  has finite total curvature and, outside some compact set,  $\Sigma$  is given by a finite collection of disjoint graphs over a common plane.

As mentioned above, Collin proved Theorem 5.2 by showing that an embedded annular end that lives in a half-space must have finite total curvature. [13] used the one-sided curvature estimate to strengthen this from a half-space to a strictly larger cone, and in the process give a very different proof of Collin's theorem.

**Theorem 5.3** (Main theorem, [13]). *There exists  $\epsilon > 0$  so that any complete properly embedded minimal annular end contained in the cone*

$$\{x_3 \geq -\epsilon(x_1^2 + x_2^2 + x_3^2)^{1/2}\} \quad (17)$$

*is asymptotic to a plane or catenoid.*

## 6. Properness and removable singularities for minimal laminations

The compactness theorems of [4]–[8] assume that the surface  $\Sigma_i$  has boundary  $\partial\Sigma_i$  in the boundary  $\partial B_{R_i}$  of an expanding sequence of balls where  $R_i$  goes to infinity. We call this the *global* case, in contrast to the *local* case where the boundaries are in the boundary of a fixed ball  $\partial B_R$ .<sup>4</sup>

This distinction between the local and global cases explains why the global compactness theorem for sequences of disks does not imply the compactness theorem for ULSC sequences. Namely, even though the ULSC sequence consists locally

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<sup>4</sup>One can also consider the more restrictive *complete* case where  $\Sigma_i$  is complete without boundary.

of disks, the compactness result for disks was in the global case where the radii go to infinity and hence does not apply.

In order to focus the discussion, we will explain the differences between the global and local cases for disks. The assumption that  $R_i \rightarrow \infty$  is used in the compactness theorem for disks in two ways:

- (1) We show that the limit lamination contains a stable leaf through each singular point. Since  $R_i \rightarrow \infty$ , this stable leaf is complete and, hence, a plane by the Bernstein theorem of D. Fischer-Colbrie and R. Schoen and M. Do Carmo and C. Peng.
- (2) We show next that the leaves nearby a singular point must also be planes. It follows that the singular set cannot stop and all of  $\mathbf{R}^3$  is foliated by planes in the limit. We call this *properness*.

The use of  $R_i \rightarrow \infty$  in (1) is not really essential. The leaf would no longer have to be flat in the local case, but it would satisfy uniform estimates by R. Schoen's curvature estimate for stable surfaces, [37] (cf. [3]).

In contrast, it turns out that the use of  $R_i \rightarrow \infty$  in (2) is essential. Namely, in [14], we constructed a sequence of embedded minimal disks  $\Sigma_i$  in the unit ball  $B_1$  with  $\partial\Sigma_i \subset \partial B_1$  where the curvatures blow up only at 0 and

$$\Sigma_i \setminus \{x_3 = 0\} \tag{18}$$

converges to two embedded minimal disks

$$\Sigma^- \subset \{x_3 < 0\} \tag{19}$$

$$\Sigma^+ \subset \{x_3 > 0\}, \tag{20}$$

each of which spirals into  $\{x_3 = 0\}$  and thus is not proper. Thus, in the example from [14], 0 is the first, last, and only point in  $\mathcal{S}_{ulsc}$  and the limit lamination consists of three leaves:  $\Sigma^+$ ,  $\Sigma^-$ , and the punctured unit disk  $B_1 \cap \{x_3 = 0\} \setminus \{0\}$ . This lamination of  $B_1 \setminus \{0\}$  cannot be extended smoothly to a lamination of  $B_1$ ; that is to say, 0 is not a removable singularity. This should be contrasted with the global case where every singular point is a removable singularity for the limit foliation by parallel planes. B. Dean has constructed similar examples where the singular set is an arbitrary finite set of points in the vertical axis; see [20].

**6.1. A sketch of the proof of properness for disks.** To explain the proof of properness in the global case for disks, we first need to see what could go wrong. Suppose therefore that the origin 0 is a singular point and  $\{x_3 = 0\}$  is the corresponding limit plane. It follows from the one-sided curvature estimate that the intersection of each  $\Sigma_j$  with a low cone about  $\{x_3 = 0\}$  consists of two multi-valued graphs for  $j$  large (the fact that there are exactly two is established in proposition II.1.3 in [7]). There are now two possibilities:

- (P) The multi-valued graphs in the complement of the cone close up in the limit to a foliation.

(N-P) These multi-valued graphs converge to a collection of graphs and *at least one* multi-valued graph that spirals infinitely on one side of  $\{x_3 = 0\}$ .

As we saw above, the second case (N-P) can occur in the local case. We will explain why it cannot happen in the global case.

Suppose therefore that (N-P) holds and the limit contains a multi-valued graph that spirals infinitely down to the plane  $\{x_3 = 0\}$ ; it is the graph of a multi-valued function  $u(\rho, \theta)$  defined for all  $\rho \geq e$  and all  $\theta > 0$ . The separation  $w(\rho, \theta)$  between consecutive sheets is by definition

$$w(\rho, \theta) = u(\rho, \theta + 2\pi) - u(\rho, \theta). \quad (21)$$

Since the limit is embedded and spirals downward, we must have  $w < 0$ . We will actually work with the conformally changed functions  $\tilde{u}(x + iy) = u(e^x, y)$  and  $\tilde{w}(x + iy) = w(e^x, y)$  that are defined on the quadrant  $\{x > 1, y > 0\}$ . The key point in the proof of properness is to show that:

(Key) The vertical flux across  $\{x = 1\}$  is negative infinity.

Why (Key) leads to a contradiction: To see why (Key) leads to a contradiction, we need to recall more about the limit in case (N-P). Namely, we showed in [7] that there must be two multi-valued graphs spiralling together just as occurs for the helicoid. The same argument applies to both multi-valued graphs, so both have unbounded negative flux across  $\{x = 1\}$ , i.e., over the circle of radius  $e$  in the plane. Moreover, we also showed in [7] that these two halves can be joined together by curves in the embedded minimal disk with a uniform bound on the length of the curves. For example, the helicoid contains two infinite valued graphs and these can be connected by horizontal lines. In any case, this leads to a flux contradiction: Stokes' theorem implies that the sum of the fluxes across compact subcurves over the circle of radius  $e$  must be bounded by the length of the connecting curves. However, the length of the connecting curves is uniformly bounded and the fluxes across the other curves both go to negative infinity.

The idea of the proof of (Key): In order to keep things simple, we will pretend that  $u$  is a harmonic function; this serves to illustrate the main ideas. Since the separation  $w$  is locally the difference of two harmonic functions,  $w$  is also harmonic; hence, the conformally changed functions  $\tilde{u}$  and  $\tilde{w}$  are harmonic on the quadrant  $\{x > 1, y > 0\}$ . Note that  $\tilde{u}$  is positive and  $\tilde{w}$  is negative.

The property (Key) is now roughly equivalent to showing that

$$\int_0^\infty \frac{\partial \tilde{u}}{\partial x}(1, y) dy = +\infty. \quad (22)$$

It may be helpful to consider an example; the function  $\tilde{u} = \pi/2 - \arctan(y/x)$  is positive, harmonic, and its multi-valued graph is an embedded infinite spiral that accumulated to the plane  $\{x_3 = 0\}$ . Furthermore, it is easy to verify (24) in this case:

$$\int_0^\infty \frac{\partial \tilde{u}}{\partial x}(1, y) dy = \int_0^\infty \frac{y}{1 + y^2} dy = +\infty. \quad (23)$$

To prove (24) for a general function  $\tilde{u}$ , first observe that Stokes' theorem gives

$$\int_1^R \frac{\partial \tilde{u}}{\partial x}(1, y) dy + \int_1^R \frac{\partial \tilde{u}}{\partial y}(x, 1) dx = \int_{\{x^2+y^2=R^2+1, x>1, y>1\}} \frac{\partial u}{\partial r}. \quad (24)$$

In [10], we used the lower bound  $\tilde{u} \geq 0$  to prove that

$$\int_{\{x^2+y^2=R^2+1, x>1, y>1\}} \frac{\partial u}{\partial r} \quad (25)$$

is essentially non-negative. The reason for this is that (25) measures the logarithmic rate of growth of the average of  $\tilde{u}$  on the semi-circle; if this was negative, the function  $\tilde{u}$  would eventually have to become negative.

We then proved the claim (24) in [10] by using a sharp estimate for the decay of  $\tilde{w}$  to show that

$$\int_1^\infty \frac{\partial \tilde{u}}{\partial y}(x, 1) dx = -\infty. \quad (26)$$

To explain this, observe that  $\tilde{w}(x, 1)$  is nothing more than  $\tilde{u}(x, 1 + 2\pi) - \tilde{u}(x, 1)$  and, hence, can be written as

$$\tilde{w}(x, 1) = \int_1^{1+2\pi} \frac{\partial \tilde{u}}{\partial y}(x, y) dy. \quad (27)$$

In particular,  $\tilde{w}(x, 1) \approx 2\pi \frac{\partial \tilde{u}}{\partial y}(x, 1)$ . We proved in [10] that the fastest possible decay for  $|\tilde{w}(x, 1)|$  is  $c_1/x$  and, consequently, we get that

$$\int_1^\infty \tilde{w}(x, 1) dx = -\infty. \quad (28)$$

This completes the sketch of the proof. The actual argument in [10] is somewhat more complicated, but similar in flavor.

## 7. The uniqueness of the helicoid

The helicoid and plane are the only classical examples of properly embedded complete minimal disks in  $\mathbf{R}^3$ . It turns out that there is a good reason for the scarcity of examples. Namely, using the compactness theorem and one-sided curvature estimate of [4]–[7], W. Meeks and H. Rosenberg proved the uniqueness of the helicoid:

**Theorem 7.1** (Main theorem, [31]). *The plane and helicoid are the only complete properly embedded simply-connected minimal surfaces in  $\mathbf{R}^3$ .*

This uniqueness has many applications, including additional regularity of the singular set  $\mathcal{S}$ . To set this up, recall that if we take a sequence of rescalings of the helicoid, then the singular set  $\mathcal{S}$  for the convergence is the vertical axis perpendicular to the leaves of the foliation. In [25], W. Meeks used this fact

together with the uniqueness of the helicoid to prove that the singular set  $\mathcal{S}$  in Theorem B.1 is always a straight line perpendicular to the foliation.

There is an analog of Theorem 7.1 in the higher genus case. Namely, any properly embedded minimal surface with finite (non-zero) genus and one end must be asymptotic to a helicoid. Until recently, it was not known whether any such surface exists; however, the construction of the genus one helicoid in [21] suggests that there may be a substantial theory of these.

**Remark 7.2.** It follows from [15] that any complete embedded minimal surface with finite topology in  $\mathbf{R}^3$  is automatically properly embedded. In particular, the hypothesis of properness can be removed from Theorem 7.1.

**7.1. A sketch of the proof.** We will give a brief overview of the proof by Meeks and Rosenberg for the uniqueness of the helicoid; we refer to the original paper [31] for the details.

The first main step in the proof of Theorem 7.1 is to analyze the asymptotic structure of a non-flat embedded minimal disk  $\Sigma$ , showing that it looks roughly like a helicoid. This is done in [31] by analyzing sequences of rescalings of  $\Sigma$ . This rescaling argument yields a sequence of embedded minimal disks which does not converge in the classical sense (there are no local area bounds). However, the lamination theorem of [4]–[7] gives that a subsequence converges to a foliation by parallel planes away from a Lipschitz curve transverse to these planes. Moreover, the lamination theorem also gives that the intersection of  $\Sigma$  with a cone consists of two asymptotically flat multi-valued graphs. In particular, this foliation is unique, i.e., does not depend on the choice of subsequence. After possibly rotating  $\mathbf{R}^3$ , we can assume that the limit foliation is by horizontal planes (i.e., level sets of  $x_3$ ).

The second main step is to show that the height function  $x_3$  together with its harmonic conjugate (which we will denote by  $x_3^*$ ) give global isothermal coordinates on  $\Sigma$ . This step is crucial since it reduces the problem to analyzing the potential Weierstrass data on the plane. There are two key components to getting these global coordinates, both of independent interest. First, one must show that  $\nabla x_3$  does not vanish on  $\Sigma$  – i.e., that the Gauss map misses the north and south poles. Second, one must show that the map  $(x_3, x_3^*)$  is proper – i.e., that  $x_3^*$  goes to infinity as we go out horizontally. Both of these steps strongly use the asymptotic structure established in the first step.

The third main step is to analyze the Weierstrass data in the conformal coordinates  $(x_3, x_3^*)$ . In these coordinates, the only unknown is a meromorphic function  $g$  that is the stereographic projection of the Gauss map of  $\Sigma$ . Since the Gauss map was already shown to miss the north and south poles, the function  $g$  can be written as  $g = e^f$  for an entire holomorphic function  $f$ . Meeks and Rosenberg then use a Picard type argument to show that  $f$  must be linear. The key in this argument is to analyze the inverse images of horizontal circles in  $\mathbf{S}^2$  under the Gauss map  $\mathbf{n}$ , using rescaling arguments and the compactness theory of [4]–[7] to control the number of components. Finally, every linear function  $f$  gives rise to a surface in the associate surface family of the helicoid, but the actual helicoid is the only one of these that is embedded.

## 8. Quasiperiodicity of properly embedded minimal planar domains

We turn next to recent results of W. Meeks, J. Perez, and A. Ros on the structure of *complete* properly embedded minimal planar domains with infinitely many ends in  $\mathbf{R}^3$ . They have obtained many important results on these surfaces in the series of papers [28]–[30]; we have chosen to focus on the main result of [28].

Many of these structure results are motivated by the two-parameter family of minimal planar domains known as the Riemann examples; see

**8.1. A few definitions.** To set this up, we first recall a few properties of a complete properly embedded minimal planar domain  $M \subset \mathbf{R}^3$ .

First, it follows from a barrier argument of Meeks and Yau that one can find a stable embedded annulus between each pair of ends of  $M$ ; a result of Fischer-Colbrie then implies that this stable surface has finite total curvature, so its ends are asymptotic to planes or half-catenoids. Since  $M$  is embedded, these planes or half-catenoids between its ends must all be parallel; this plane is the *limit tangent plane at infinity*.

Using these planes and half-catenoids in this way, Callahan, Frohman, Hoffman, and Meeks showed that the ends of  $M$  are ordered by height over this limit tangent plane at infinity. Moreover, a nice argument of Collin, Kusner, Meeks, and Rosenberg in [19] shows that there are at most two *limit ends*; these can only be on the “top” or on the “bottom”.

Finally, since the coordinate functions are harmonic on any minimal surface, it follows from Stokes’ theorem that the flux of a coordinate function  $x_i$  around a closed curve  $\gamma$  depends only on its homology class  $[\gamma]$ . Recall that the flux of  $x_i$  around  $\gamma$  is

$$\int_{\gamma} \frac{\partial x_i}{\partial n}, \tag{29}$$

where  $\frac{\partial x_i}{\partial n}$  is the derivative of  $x_i$  in the conormal direction, i.e., the direction tangent to  $M$  but normal to  $\gamma$ . Using all three coordinates functions at once gives the *flux* map

$$Flux : [\gamma] \rightarrow \mathbf{R}^3. \tag{30}$$

**8.2. The curvature estimate of [28].** We can now define  $\mathcal{M}$  to be the space of properly embedded minimal planar domains in  $\mathbf{R}^3$  with two limit ends, normalized so that every surface  $M \in \mathcal{M}$  has horizontal limit tangent plane at infinity and the vertical component of its flux equals one. Here the horizontal flux is the projection of the flux vector to the limit tangent plane at infinity.

The main theorem of [28] is:

**Theorem 8.1** (Theorem 5, [28]). *If a sequence  $M_i \in \mathcal{M}$  has bounded horizontal flux, then the Gaussian curvature of the sequence is uniformly bounded.*

**Remark 8.2.** The main result in [29] is that there must always be two limit ends; thus this hypothesis can be removed from Theorem 8.1.

The first important application of the curvature estimates is to describe the geometry of a properly embedded minimal planar domain  $M$  with two limit ends:

Such a surface  $M$  has bounded curvature and is conformally a compact Riemann surface punctured in a countable closed subset with two limit points; the spacing between consecutive ends is bounded from below in terms of the bound for the curvature;  $M$  is quasiperiodic in the sense that there exists a divergent sequence  $V(n) \in \mathbb{R}^3$  such that the translated surfaces  $M+V(n)$  converge to a properly embedded minimal surface of genus zero, two limit ends, a horizontal limit tangent plane at infinity and with the same flux as  $M$ .

Another particularly interesting consequence of Theorem 8.1 is a solution of an old conjecture of Nitsche:

**Theorem 8.3** (Theorem 1, [28]). *Any complete minimal surface which is a union of simple closed curves in horizontal planes must be a catenoid.*

**8.3. A heuristic argument for Theorem 8.1.** Theorem 8.1 is best illustrated by considering the family of singly-periodic minimal surfaces known as the Riemann examples. After normalizing so the vertical flux is one and rotating so the horizontal flux points in the  $x_1$ -direction, there is a 1-parameter family parameterized by the length of the horizontal flux  $H$ . As  $H \rightarrow 0$ , this degenerates to a catenoid; when  $H \rightarrow \infty$ , this degenerates to a helicoid.

With this in mind, there is a simple heuristic argument for why such an estimate should hold. Namely, assume that  $|A|^2(0) \rightarrow \infty$  for a sequence  $\Sigma_i$  and consider two possibilities:

1. The injectivity radius goes to zero.
2. Each  $\Sigma_i$  is uniformly simply connected.

In the first case, we would get short dividing curves in  $\Sigma_i$ ; integrating around these would then imply that the flux was going to zero (violating the normalization).

In the second case, the results of Colding and the author in [8] give a limit plane through the origin which, by the uniqueness of the helicoid of Meeks and H. Rosenberg, is locally modelled by the helicoid for large  $i$ . As discussed above, this corresponds (roughly, at least) to a great deal of horizontal flux, violating the assumed bound on this.

The actual argument is much more complicated but has at least a similar flavor. One reason for the complications is that the dichotomy (1) or (2) above is more subtle; (1) involves the intrinsic distance while (2) uses the extrinsic distance. This dichotomy can now be made rigorous using the proof of the Calabi-Yau conjectures for embedded minimal surfaces with finite topology in [15]; however, the proof of Theorem 8.1 came before [15], so intrinsic and extrinsic distances were not yet known to be equivalent.

## A. Multi-valued graphs

We have used two notions of multi-valued graphs - namely, the one used in [4]–[7] and a generalization.

In [4]–[7], we defined multi-valued graphs as multi-sheeted covers of the punctured plane. To be precise, let  $D_r$  be the disk in the plane centered at the origin and of radius  $r$  and let  $\mathcal{P}$  be the universal cover of the punctured plane  $\mathbf{C} \setminus \{0\}$  with global polar coordinates  $(\rho, \theta)$  so  $\rho > 0$  and  $\theta \in \mathbf{R}$ . An  $N$ -valued graph of a function  $u$  on the annulus  $D_s \setminus D_r$  is a single valued graph over

$$\{(\rho, \theta) \mid r \leq \rho \leq s, |\theta| \leq N\pi\}. \quad (31)$$

Note that the helicoid is the union of two infinite-valued graphs over the punctured plane together with the vertical axis.

Locally, the above multi-valued graphs give the complete picture for a ULSC sequence. However, the global picture can consist of several different multi-valued graphs glued together. To allow for this, we are forced to consider multi-valued graphs defined over the universal cover of  $\mathbf{C} \setminus P$  where  $P$  is a discrete subset of the complex plane  $\mathbf{C}$ . We will see that the bound on the genus implies that  $P$  consists of at most two points.

## B. The lamination theorem and one-sided curvature estimate

The first theorem that we recall shows that embedded minimal disks are either graphs or are part of double spiral staircases; moreover, a sequence of such disks with curvature blowing up converges to a foliation by parallel planes away from a singular curve  $\mathcal{S}$ . This theorem is modelled on rescalings of the helicoid and the precise statement is as follows (we state the version for extrinsic balls; it was extended to intrinsic balls in [11]):

**Theorem B.1.** *(Theorem 0.1 in [7].) Let  $\Sigma_i \subset B_{R_i} = B_{R_i}(0) \subset \mathbf{R}^3$  be a sequence of embedded minimal disks with  $\partial\Sigma_i \subset \partial B_{R_i}$  where  $R_i \rightarrow \infty$ . If*

$$\sup_{B_1 \cap \Sigma_i} |A|^2 \rightarrow \infty, \quad (32)$$

*then there exists a subsequence,  $\Sigma_j$ , and a Lipschitz curve  $\mathcal{S} : \mathbf{R} \rightarrow \mathbf{R}^3$  such that after a rotation of  $\mathbf{R}^3$ :*

- 1.  $x_3(\mathcal{S}(t)) = t$ . (That is,  $\mathcal{S}$  is a graph over the  $x_3$ -axis.)*
- 2. Each  $\Sigma_j$  consists of exactly two multi-valued graphs away from  $\mathcal{S}$  (which spiral together).*
- 3. For each  $1 > \alpha > 0$ ,  $\Sigma_j \setminus \mathcal{S}$  converges in the  $C^\alpha$ -topology to the foliation,  $\mathcal{F} = \{x_3 = t\}_t$ , of  $\mathbf{R}^3$ .*

4.  $\sup_{B_r(\mathcal{S}(t)) \cap \Sigma_j} |A|^2 \rightarrow \infty$  for all  $r > 0$ ,  $t \in \mathbf{R}$ . (The curvatures blow up along  $\mathcal{S}$ .)

The second theorem that we need to recall asserts that every embedded minimal disk lying above a plane, and coming close to the plane near the origin, is a graph. Precisely this is the *intrinsic one-sided curvature estimate* which follows by combining [7] and [11]:

**Theorem B.2.** *There exists  $\epsilon > 0$ , so that if*

$$\Sigma \subset \{x_3 > 0\} \subset \mathbf{R}^3 \quad (33)$$

*is an embedded minimal disk with  $\mathcal{B}_{2R}(x) \subset \Sigma \setminus \partial\Sigma$  and  $|x| < \epsilon R$ , then*

$$\sup_{\mathcal{B}_R(x)} |A_\Sigma|^2 \leq R^{-2}. \quad (34)$$

Theorem B.2 is in part used to prove the regularity of the singular set where the curvature is blowing up.

Note that the assumption in Theorem B.1 that the surfaces are disks is crucial and cannot even be replaced by assuming that the sequence is ULSC. To see this, observe that one can choose a one-parameter family of Riemann examples which is ULSC but where the singular set  $\mathcal{S}$  is given by a pair of vertical lines. Likewise, the assumption in Theorem B.2 that  $\Sigma$  is simply connected is crucial as can be seen from the example of a rescaled catenoid, see (3). Under rescalings the catenoid converges (with multiplicity two) to the flat plane. Thus a neighborhood of the neck can be scaled arbitrarily close to a plane but the curvature along the neck becomes unbounded as it gets closer to the plane. Likewise, by considering the universal cover of the catenoid, one sees that embedded, and not just immersed, is needed in Theorem B.2.

## C. The precise statement of the general compactness theorem

The precise statement of the compactness theorem for sequences that are neither necessarily ULSC nor with  $\mathcal{S}_{ulsc} = \emptyset$  is the following:

**Theorem C.1** ([8]). *Let  $\Sigma_i \subset B_{R_i} = B_{R_i}(0) \subset \mathbf{R}^3$  be a sequence of compact embedded minimal planar domains with  $\partial\Sigma_i \subset \partial B_{R_i}$  where  $R_i \rightarrow \infty$ . If*

$$\sup_{B_1 \cap \Sigma_i} |A|^2 \rightarrow \infty, \quad (35)$$

*then there is a subsequence  $\Sigma_j$ , a closed set  $\mathcal{S}$ , and a lamination  $\mathcal{L}'$  of  $\mathbf{R}^3 \setminus \mathcal{S}$  so that:*

(A) *For each  $1 > \alpha > 0$ ,  $\Sigma_j \setminus \mathcal{S}$  converges in the  $C^\alpha$ -topology to the lamination  $\mathcal{L}'$ .*

(B)  $\sup_{B_r(x) \cap \Sigma_j} |A|^2 \rightarrow \infty$  as  $j \rightarrow \infty$  for all  $r > 0$  and  $x \in \mathcal{S}$ . (The curvatures blow up along  $\mathcal{S}$ .)

(C1)  $(C_{neck})$  from Theorem 4.5 holds for each point  $y$  in  $\mathcal{S}_{neck}$ .

(C2)  $(C_{ulsc})$  from Theorem 4.3 holds locally near  $\mathcal{S}_{ulsc}$ . More precisely, each point  $y$  in  $\mathcal{S}_{ulsc}$  comes with a sequence of multi-valued graphs in  $\Sigma_j$  that converge to the plane  $\{x_3 = x_3(y)\}$ . The convergence is in the  $C^\infty$  topology away from the point  $y$  and possibly also one other point in  $\{x_3 = x_3(y)\} \cap \mathcal{S}_{ulsc}$ . These two possibilities correspond to the two types of multi-valued graphs defined in Section A.

(D) The set  $\mathcal{S}_{ulsc}$  is a union of Lipschitz curves transverse to the lamination. The leaves intersecting  $\mathcal{S}_{ulsc}$  are planes foliating an open subset of  $\mathbf{R}^3$  that does not intersect  $\mathcal{S}_{neck}$ . For the set  $\mathcal{S}_{neck}$ , we make no claim about the structure.

(P) Together (C1) and (C2) give a sequence of graphs or multi-valued graphs converging to a plane through each point of  $\mathcal{S}$ . If  $P$  is one of these planes, then each leaf of  $\mathcal{L}'$  is either disjoint from  $P$  or is contained in  $P$ .

Note that Theorem C.1 is a technical tool that is superseded by the stronger compactness theorems in the ULSC and non-ULSC cases, Theorem 4.3 and Theorem 4.5. This is because we will know by the no mixing theorem that either  $\mathcal{S}_{neck} = \emptyset$  or  $\mathcal{S}_{ulsc} = \emptyset$ , so that these cover all possible cases.

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