

# SHARP ESTIMATES FOR MEAN CURVATURE FLOW OF GRAPHS

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## 0. INTRODUCTION

A one-parameter family of smooth hypersurfaces  $\{M_t\} \subset \mathbf{R}^{n+1}$  flows by mean curvature if

$$z_t = \mathbf{H}(z) = \Delta_{M_t} z, \tag{0.1}$$

where  $z$  are coordinates on  $\mathbf{R}^{n+1}$  and  $\mathbf{H} = -H\mathbf{n}$  is the mean curvature vector.

In this note, we prove sharp gradient and area estimates for graphs flowing by mean curvature. Thus, each  $M_t$  is assumed to be the graph of a function  $u(\cdot, t)$ . So, if  $z = (x, y)$  with  $x \in \mathbf{R}^n$ , then  $M_t$  is given by  $y = u(x, t)$ . Below,  $du$  is the  $\mathbf{R}^n$  gradient of a function  $u$ ,  $\|u\|_\infty$  is the sup norm, and  $B_s$  is the ball in  $\mathbf{R}^n$  with radius  $s$  centered at the origin.

Our gradient estimate is the following (see Section 2 for the sharp area estimate):

**Theorem 1.** There exists  $C = C(n)$  so if the graph of  $u : B_{\sqrt{2n+1}r} \times [0, r^2] \rightarrow \mathbf{R}$  flows by mean curvature, then

$$\log |du|(0, r^2/[4n]) \leq C (1 + r^{-1} \|u(\cdot, 0)\|_\infty)^2. \tag{0.2}$$

The quadratic dependence on  $\|u(\cdot, 0)\|_\infty$  in (0.2) should be compared with the linear dependence which holds when the graph of  $u$  is minimal (i.e.,  $u_t = 0$ ). In the minimal case, Bombieri, De Giorgi, and Miranda proved in [BDM] that

$$\log |du|(0) \leq C (1 + r^{-1} \|u\|_\infty) \tag{0.3}$$

(the case of surfaces was done by Finn in [F1]). By an earlier example of Finn, this exponential dependence cannot be improved even in the minimal case (see [F2] and cf. [GiTr]).

In [K], Korevaar gave a maximum principle proof of a weaker form of [BDM]; this weaker form had  $\|u\|_\infty^2$  in place of  $\|u\|_\infty$ . Ecker and Huisken adapted Korevaar's argument to mean curvature flow in theorem 2.3 of [EH2] to get

$$\log |du|(0, r^2/[4n]) \leq 1/2 \log (1 + \|du(\cdot, 0)\|_\infty^2) + C (1 + r^{-1} \|u(\cdot, 0)\|_\infty)^2. \tag{0.4}$$

Note that, unlike (0.2), the gradient bound (0.4) depends also on the initial bound for the gradient.

Using the so-called grim reaper, one can see that the quadratic dependence on  $\|u(\cdot, 0)\|_\infty$  in Theorem 1 is sharp (see Proposition 1 below). The *grim reaper* is the translating solution to the mean curvature flow given by that for each  $t$  it is a graph of the function

$$u(x, t) = t - \log \sin x, \tag{0.5}$$

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The authors were partially supported by NSF Grants DMS 0104453 and DMS 0104187.

where  $x \in (0, \pi)$  and  $t \in [0, \infty)$ . Note that  $-u(x, t)$  is a downward translating solution. More generally, a parabolic rescaling by  $\lambda > 0$  gives that the graph of

$$u^\lambda(x, t) = \frac{1}{\lambda}u(\lambda x, \lambda^2 t) = \lambda t - \frac{\log \sin(\lambda x)}{\lambda}, \quad (0.6)$$

where  $x \in (0, \pi/\lambda)$ , is a translating solution flowing with speed  $\lambda$ ; see figure 1. Since  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ , an easy calculation shows that for  $\lambda > 0$  sufficiently large

$$u^\lambda(e^{-\lambda^2}, 1) = \lambda - \frac{\log \sin(\lambda e^{-\lambda^2})}{\lambda} \leq 2\lambda. \quad (0.7)$$

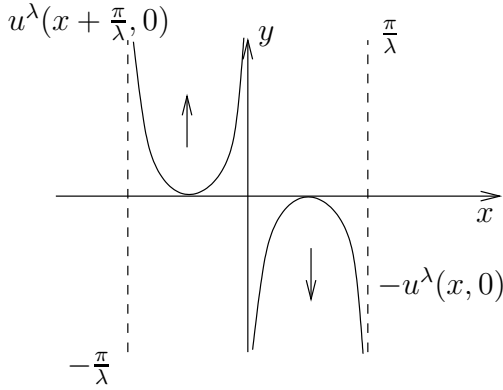


FIGURE 1. Two scaled grim reapers.

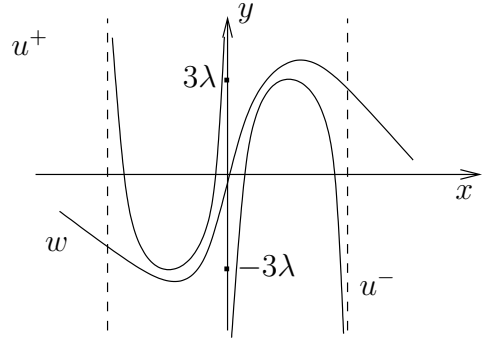


FIGURE 2. Use  $u^+ = u^\lambda(x + \frac{\pi}{\lambda}, t) - 3\lambda$  and  $u^- = -u^\lambda(x, t) + 3\lambda$  as barriers and let  $w$  be a graph between  $u^+$  and  $u^-$ .

**Proposition 1.** Given  $\lambda > 1$  sufficiently large, there is a solution  $w(x, t)$  on  $\mathbf{R} \times [0, \infty)$  of the mean curvature flow with

$$3\lambda < \|w(\cdot, 0)\|_\infty \leq 4\lambda, \quad (0.8)$$

$$\lambda e^{\lambda^2} \leq \max_{|x| \leq e^{-\lambda^2}} |dw(x, 1)|. \quad (0.9)$$

*Proof.* Define solutions  $u^+(x, t) = u^\lambda(x + \pi/\lambda, t) - 3\lambda$  for  $-\pi/\lambda < x < 0$  and  $u^-(x, t) = -u^\lambda(x, t) + 3\lambda$  for  $0 < x < \pi/\lambda$  of the mean curvature flow to be used as barriers. Since  $u^+ \geq -3\lambda$  and  $u^- \leq 3\lambda$ , it is easy to choose (see figure 2) a smooth compactly supported function  $w(\cdot, 0) : \mathbf{R} \rightarrow \mathbf{R}$  satisfying (0.8) and so

$$w(x, 0) < u^+(x, 0) \text{ for } -\pi/\lambda < x < 0, \quad (0.10)$$

$$u^-(x, 0) < w(x, 0) \text{ for } 0 < x < \pi/\lambda. \quad (0.11)$$

(We can choose  $w(\cdot, 0)$  so that  $w(x, 0) = 0$  for  $|x| > \pi/\lambda$ .) The existence results of [EH1] or [EH2] (see, e.g., theorem 1.7 in [E]) extend  $w(x, 0)$  to a solution  $w(x, t)$  of the mean curvature flow defined for  $x \in \mathbf{R}$  and  $t \in [0, \infty)$ ; see figure 3. Moreover, the maximum principle extends (0.10) and (0.11) to all  $t \geq 0$ . In particular, using this at  $x = \pm e^{-\lambda^2}$ ,  $t = 1$

and substituting (0.7), we get that

$$w(-e^{-\lambda^2}, 1) < u^+(-e^{-\lambda^2}, 1) \leq -\lambda, \quad (0.12)$$

$$\lambda \leq u^-(e^{-\lambda^2}, 1) < w(e^{-\lambda^2}, 1). \quad (0.13)$$

Finally, combining (0.12), (0.13), and the mean value theorem gives (0.9); see figure 3.  $\square$

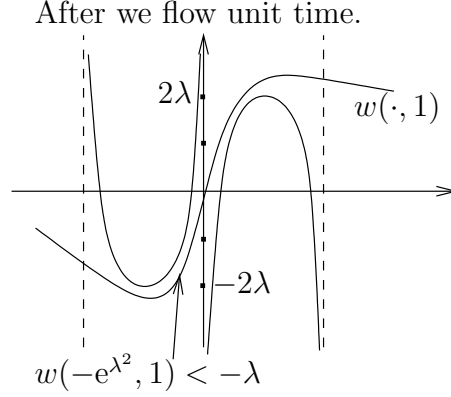


FIGURE 3. By the maximum principle,  $w(\cdot, 1)$  will be between the barriers. This bounds  $\max |\nabla w|$  from below.

Throughout,  $\nabla$ ,  $\Delta$ , and  $\mathbf{n}$  are the induced covariant derivative, laplacian, and unit normal on the submanifold  $M_t$  of  $\mathbf{R}^{n+1}$ . The graph of a function  $u$  flows by mean curvature if

$$u_t = (1 + |du|^2)^{1/2} \operatorname{div} \left( \frac{du}{(1 + |du|^2)^{1/2}} \right), \quad (0.14)$$

where  $\operatorname{div}$  is divergence in  $\mathbf{R}^n$ .

## 1. INTERIOR GRADIENT ESTIMATE FOR GRAPHS

Theorem 1 will follow immediately from the next proposition and a standard maximum principle bounding  $u$  at future times in terms of the initial bound, see Lemma 3.

**Proposition 2.** If the graph of  $u : B_r \times [0, r^2] \rightarrow \mathbf{R}$  flows by mean curvature, then

$$\log(1 + |du|^2(0, r^2/[4n])) \leq 2 \log 10 + 16n(1 + 2r^{-1} \|u\|_\infty)^2. \quad (1.1)$$

The strategy of the proof of Proposition 2 is as follows. By the maximum principle (Lemma 1) at the maximum of  $\phi v$  (where  $\phi$  is a cutoff function and  $v = (1 + |du|^2)^{1/2}$ ) the heat operator of the cutoff function is nonnegative. By choosing an appropriate cutoff function in terms of  $u$  (Lemma 2), we can bound the heat operator of the cutoff from above in terms of the gradient of  $u$ . Playing off this lower and upper bound at the max against each other gives the proposition.

**Lemma 1.** Suppose  $(\partial_t - \Delta)v \leq -2|\nabla v|^2/v$  for a function  $v \geq 0$  on  $\{M_t\}_{t \in [0, 1]}$ . If the function  $\phi$  is  $\leq 0$  on  $M_0 \cup_t (\partial M_t)$  but  $\max \phi v > 0$ , then  $(\partial_t - \Delta)\phi \geq 0$  at the maximum of  $\phi v$ .

*Proof.* At the maximum of  $\phi v$ , we get

$$\nabla(\phi v) = v\nabla\phi + \phi\nabla v = 0, \quad (1.2)$$

$$(\partial_t - \Delta)(\phi v) = v\partial_t\phi + \phi\partial_tv - v\Delta\phi - 2\langle\nabla\phi, \nabla v\rangle - \phi\Delta v \geq 0. \quad (1.3)$$

Substituting (1.2) into (1.3) and using  $(\partial_t - \Delta)v \leq -2|\nabla v|^2/v$  gives  $(\partial_t - \Delta)\phi \geq 0$ .  $\square$

We will apply Lemma 1 to the volume element  $v = (1 + |du|^2)^{1/2}$ . We will first need some elementary formulas. If the graph of  $u$  flows by mean curvature and  $a \in \mathbf{R}$ , then

$$(\partial_t - \Delta)e^{ay^2/t} = -\frac{e^{ay^2/t}}{t^2} [ay^2 + 4a^2y^2|\nabla y|^2 + 2at|\nabla y|^2], \quad (1.4)$$

and (see, e.g., lemma 1.1 in [EH2] or 2.11 in [E])

$$(\partial_t - \Delta)v = -|A|^2v - 2\frac{|\nabla v|^2}{v} \leq -2\frac{|\nabla v|^2}{v}, \quad (1.5)$$

$$(\partial_t - \Delta)(1 - |x|^2 - 2nt) \leq 0. \quad (1.6)$$

The next lemma introduces the cutoff  $\phi$  which will be used in Lemma 1.

**Lemma 2.** Set  $\phi = \eta e^{ay^2/t}$  for  $a \in \mathbf{R}$  and  $\eta = (1 - |x|^2 - 2nt)$ . If the graph of  $u : B_1 \times [0, 1] \rightarrow \mathbf{R}$  flows by mean curvature, then

$$(\partial_t - \Delta)\phi \leq -\frac{e^{ay^2/t}}{t^2} \left[ ay^2\eta + (4a^2y^2 + 2at)\eta\frac{|du|^2}{1 + |du|^2} - 8|ay|t\frac{|du|}{1 + |du|^2} \right]. \quad (1.7)$$

*Proof.* Using (1.4) and (1.6) gives

$$\begin{aligned} (\partial_t - \Delta)\phi &= \eta(\partial_t - \Delta)e^{ay^2/t} + e^{ay^2/t}(\partial_t - \Delta)\eta - 4e^{ay^2/t}\frac{ay}{t}\langle\nabla\eta, \nabla y\rangle \\ &\leq -\frac{e^{ay^2/t}}{t^2} [(ay^2 + 4a^2y^2|\nabla y|^2 + 2at|\nabla y|^2)\eta - 4ayt\langle\nabla y, \nabla|x|^2\rangle]. \end{aligned} \quad (1.8)$$

The lemma follows since  $|x| \leq 1$  and the  $y$  component of the normal is  $(1 + |du|^2)^{-1/2}$ .  $\square$

*Proof.* (of Proposition 2.) By scaling, it suffices to prove the proposition when  $r = 1$ . Set  $\eta = (1 - |x|^2 - 2nt)$  and  $\phi = \eta e^{ay^2/t}$  for  $a \leq -2$  to be chosen. After replacing  $u$  by  $u + \|u\|_\infty + 1$  (i.e., translating), we can assume that  $u \geq 1$ ; in particular,  $\phi$  vanishes when  $t = 0$ . If the maximum of  $\phi v$  for  $t \in [0, 1]$  is at  $(x_0, y_0, t_0) \in B_1 \times \mathbf{R} \times (0, 1]$ , then (1.5) together with Lemmas 1 and 2 give

$$ay_0^2\eta + (4a^2y_0^2 + 2at_0)\eta\frac{|du|^2(x_0, t_0)}{1 + |du|^2(x_0, t_0)} - 8|ay_0|t_0\frac{|du|(x_0, t_0)}{1 + |du|^2(x_0, t_0)} \leq 0. \quad (1.9)$$

There are now two cases. Namely, either

$$|ay_0|\eta|du|(x_0, t_0) < 8, \quad (1.10)$$

or  $|ay_0|\eta|du|(x_0, t_0) \geq 8$ ; in the second case, (1.9) (and  $4a^2y_0^2 + 2at_0 > 2a^2y_0^2$ ) yields

$$ay_0^2\eta + a^2y_0^2\eta\frac{|du|^2(x_0, t_0)}{1 + |du|^2(x_0, t_0)} \leq 0. \quad (1.11)$$

Since  $\eta \leq 1$ , we get in either case that

$$\eta|du|(x_0, t_0) \leq 4, \quad (1.12)$$

Since  $\max_{[0,1]}(\phi v) = \phi v(x_0, y_0, t_0)$  and  $a < 0$ , we get

$$\phi v = \eta e^{ay^2/t} (1 + |du|^2)^{1/2} \leq 5. \quad (1.13)$$

□

A standard barrier argument using shrinking spheres bounds the future height by the initial height:

**Lemma 3.** If  $\rho \geq (2n+1)^{1/2}$  and the graph of  $u : B_{\rho r} \times [0, r^2] \rightarrow \mathbf{R}$  flows by mean curvature, then

$$\begin{aligned} \max_{B_r \times [0, r^2]} |u(x, t)| &\leq r \left[ \rho - (\rho^2 - (2n+1))^{1/2} \right] + \max_{B_{\rho r}} |u(x, 0)| \\ &\leq \frac{(2n+1)r}{\rho} + \max_{B_{\rho r}} |u(x, 0)|. \end{aligned} \quad (1.14)$$

*Proof.* By scaling, it suffices to prove the lemma when  $r = 1$ . Recall that the one-parameter family  $M_t$  of concentric spheres in  $\mathbf{R}^{n+1}$  of radius  $(\rho^2 - 2nt)^{1/2}$  centered at  $x = 0, y = \rho + \max_{B_\rho} u(x, 0) + \epsilon$  is a solution to mean curvature flow. For  $\epsilon > 0$ ,  $M_0$  does not intersect the graph of  $u(\cdot, 0)$ . Applying the maximum principle and letting  $\epsilon \rightarrow 0$ , we get that

$$\max_{B_1 \times [0, 1]} u(x, t) \leq \max_{B_\rho} u(x, 0) + \rho - (\rho^2 - (2n+1))^{1/2}. \quad (1.15)$$

This, and a similar argument for the minimum of  $u$ , gives (1.14). □

## 2. AREA ESTIMATES FOR GRAPHS FLOWING BY MEAN CURVATURE

In this section, we prove an area bound for graphs flowing by mean curvature which depends quadratically on the  $L^\infty$  norm of the initial height (integrating our gradient estimate gives an exponential bound). We also give an example showing that this is sharp.

**Theorem 2.** There exists  $C = C(n)$  so if the graph of  $u : B_{\sqrt{2n+1}r} \times [0, r^2] \rightarrow \mathbf{R}$  flows by mean curvature, then

$$\text{Area}(u(B_{r/2}, r^2)) = \int_{B_{r/2}} (1 + |du|^2)^{1/2}(x, r^2) dx \leq C r^n (1 + r^{-1} \|u(\cdot, 0)\|_\infty)^2. \quad (2.1)$$

Before proving Theorem 2, we first argue as in Proposition 1 to see that the quadratic dependence on  $\|u(\cdot, 0)\|_\infty$  is sharp:

**Proposition 3.** Given an integer  $k > 1$ , there is a solution  $w(x, t)$  on  $\mathbf{R} \times [0, \infty)$  of the mean curvature flow with

$$2k < \|w(\cdot, 0)\|_\infty \leq 3k, \quad (2.2)$$

$$4k^2 - 2k \leq \int_{-\pi}^{\pi} (1 + |dw(x, 1)|^2)^{1/2} dx. \quad (2.3)$$

*Proof.* For  $-k \leq j \leq k$ , define translating solutions  $u_j$  on  $j\pi/k < x < (j+1)\pi/k$  by

$$u_j(x, t) = (-1)^j [u^k(x - j\pi/k, t) - 2k], \quad (2.4)$$

where  $u^k$  is the scaled grim reaper. The solutions given by (2.4), which alternate between translating up and down, will be used as barriers; see figure 4. As in the proof of Proposition

1, we can choose a compactly supported function  $w(\cdot, 0) : \mathbf{R} \rightarrow \mathbf{R}$  satisfying (2.2) which is below the upward translating solutions and above the downward translating solutions. Combining the existence results of [EH1] or [EH2] with the maximum principle as before gives a solution  $w(x, t)$  with

$$w((j + 1/2)\pi/k, 1) < u_j((j + 1/2)\pi/k, 1) = -k \text{ for } j \text{ even}, \quad (2.5)$$

$$k = u_j((j + 1/2)\pi/k, 1) < w((j + 1/2)\pi/k, 1) \text{ for } j \text{ odd}. \quad (2.6)$$

The lower bound on length in (2.3) follows immediately.  $\square$

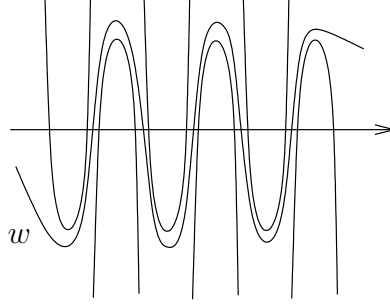


FIGURE 4. Alternating rescaled grim reapers.

We will prove Theorem 2 by showing that the (weighted) area of the graph satisfies a differential inequality which will imply the desired bound (see Lemma 5).

We begin with an elementary area bound for the graph of a general function  $w$ :

**Lemma 4.** If  $w, \phi : \mathbf{R}^n \rightarrow \mathbf{R}$  are functions and  $\phi$  has compact support, then

$$\int \phi^2 (1 + |dw|^2)^{1/2} dx \leq \int \phi^2 dx + \|w\|_\infty \int |d\phi^2| dx + \|w\|_\infty \int \phi^2 |H| dx, \quad (2.7)$$

where  $H = -\operatorname{div} \left( \frac{dw}{(1+|dw|^2)^{1/2}} \right)$  is the mean curvature of the graph of  $w$ .

*Proof.* Applying Stokes theorem to  $\operatorname{div} \left( \frac{\phi^2 w dw}{(1+|dw|^2)^{1/2}} \right)$  gives

$$\int \phi^2 \frac{|dw|^2}{(1+|dw|^2)^{1/2}} dx \leq \int |d\phi^2| \frac{|w| |dw|}{(1+|dw|^2)^{1/2}} dx + \int \phi^2 |w| |H| dx. \quad (2.8)$$

Adding  $\int \phi^2 dx$  to each side gives (2.7).  $\square$

When the graph of  $w$  is minimal (i.e.,  $H = 0$ ), Lemma 4 gives the well-known area bound  $C r^n (1 + r^{-1} \|w\|_\infty)$ . This linear dependence on  $\|w\|_\infty$  is easily seen to be sharp.

**Lemma 5.** If  $f(t) \geq 0$  and  $f^2 \leq -a f' + b$  with  $a, b > 0$ , then  $f(T) \leq \sqrt{2b} + 2a/T$ .

*Proof.* If  $f^2(T) < 2b$ , then we are done. If  $f^2(t) \geq 2b$ , then  $f^2(t) \leq -2a f'(t)$  so

$$(1/f)'(t) = -f'(t)/f^2(t) \geq 1/(2a). \quad (2.9)$$

In particular, if  $f^2(t) \geq 2b$  on  $[t_0, T]$ , then

$$1/[f(T)] \geq 1/[f(t_0)] + (T - t_0)/2a. \quad (2.10)$$

We consider two cases. First, if  $f^2(t) > 2b$  on  $[0, T]$ , then (2.10) yields  $f(T) \leq 2a/T$ . Otherwise, if  $f(t_0) = \sqrt{2b}$  for some  $t_0 < T$ , then (2.10) gives  $f(T) \leq f(t_0) = \sqrt{2b}$ .  $\square$

*Proof.* (of Theorem 2). By scaling, we can assume that  $r = 1$ . Within this proof, we write  $\|u\|_\infty$  for the  $L^\infty$  norm of  $u$  on  $B_1 \times [0, 1]$ . Set  $\eta(x) = \max\{(1 - |x|), 0\}$  and define

$$f(t) = \int \eta^4 (1 + |du|^2)^{1/2}(x, t) dx. \quad (2.11)$$

(We will omit the  $(x, t)$  below.) Differentiating  $f(t)$  and using Stokes theorem gives

$$f'(t) = \int \eta^4 \frac{\langle du, du_t \rangle}{(1 + |du|^2)^{1/2}} dx = - \int \eta^4 H^2 (1 + |du|^2)^{1/2} dx + 4 \int H \eta^3 \langle d\eta, du \rangle dx. \quad (2.12)$$

The absorbing inequality  $4|H| \eta^3 \leq H^2 \eta^4 / 2 + 8\eta^2$  then gives

$$\int \eta^4 H^2 (1 + |du|^2)^{1/2} dx \leq -2f'(t) + 16 \int \eta^2 (1 + |du|^2)^{1/2} dx. \quad (2.13)$$

Applying Lemma 4 with  $\phi = \eta$  and using an absorbing inequality gives

$$16 \int \eta^2 (1 + |du|^2)^{1/2} dx \leq C_1 (1 + \|u\|_\infty) + C_2 \|u\|_\infty^2 + 1/2 \int \eta^4 H^2 dx. \quad (2.14)$$

Combining (2.13) and (2.14) gives

$$\int \eta^4 H^2 (1 + |du|^2)^{1/2} dx \leq -4f'(t) + C_3 (1 + \|u\|_\infty^2). \quad (2.15)$$

After applying Lemma 4 with  $\phi = \eta^2$ , the Cauchy-Schwarz inequality and (2.15) give

$$\begin{aligned} f^2(t) &\leq C_4 \left( 1 + \|u\|_\infty^2 + \|u\|_\infty^2 \left( \int \eta^4 H^2 dx \right) \left( \int \eta^4 dx \right) \right) \\ &\leq C_4 (1 + \|u\|_\infty^2) + C_5 \|u\|_\infty^2 (-4f'(t) + C_3 (1 + \|u\|_\infty^2)). \end{aligned} \quad (2.16)$$

Finally, applying Lemma 5 gives the theorem since, by Lemma 3,  $\|u\|_\infty \leq \sqrt{2n+1} + \sup_{B_{\sqrt{2n+1}}} |u(\cdot, 0)|$ .  $\square$

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