THE SPACE OF EMBEDDED MINIMAL SURFACES OF FIXED GENUS IN A 3-MANIFOLD II; MULTI-VALUED GRAPHS IN DISKS

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0. Introduction

This paper is the second in a series where we attempt to give a complete description of the space of all embedded minimal surfaces of fixed genus in a fixed (but arbitrary) closed 3-manifold. The key for understanding such surfaces is to understand the local structure in a ball and in particular the structure of an embedded minimal disk in a ball in $\mathbf{R}^3$. We show here that if the curvature of such a disk becomes large at some point, then it contains an almost flat multi-valued graph nearby that continues almost all the way to the boundary.

Let $\mathcal{P}$ be the universal cover of the punctured plane $\mathbf{C}\setminus\{0\}$ with global (polar) coordinates (ρ,θ). An N-valued graph Σ over the annulus $D_{r_2}\setminus D_{r_1}$ is a (single-valued) graph over

$$\{(\rho,\theta)\in\mathcal{P}\,|\,r_1<\rho<r_2 \text{ and } |\theta|\leq \pi\,N\}\,. \tag{0.1}$$

Theorem 0.2. Given $N\in\mathbf{Z}_+$, $\epsilon>0$, there exist C_1, $C_2>0$ so: Let $0\in\Sigma^2\subset B_R\subset\mathbf{R}^3$ be an embedded minimal disk, $\partial\Sigma\subset\partial B_R$. If $\max_{B_{r_0}\cap\Sigma}|A|^2\geq 4\,C_1^2\,r_0^{-2}$ for some $R>r_0>0$, then there exists (after a rotation) an N-valued graph $\Sigma_g\subset\Sigma$ over $D_{R/C_2}\setminus D_{2r_0}$ with gradient $\leq\epsilon$ and $\Sigma_g\subset\{x_3^2\leq\epsilon^2\,(x_1^2+x_2^2)\}$.

This theorem is modeled by one half of the helicoid and its rescalings. Recall that the helicoid is the minimal surface Σ^2 in $\mathbf{R}^3$ parameterized by

$$(s\,\cos t, s\sin t, t) \tag{0.3}$$

where $s,t\in\mathbf{R}$. By one half of the helicoid we mean the multi-valued graph given by requiring that $s>0$ in (0.3).

Theorem 0.2 will follow by combining a blow up result with [CM3]. This blow up result says that if an embedded minimal disk in a ball has large curvature at a point, then it contains a small almost flat multi-valued graph nearby, that is:

Theorem 0.4. Given $N,\omega>1$, and $\epsilon>0$, there exists $C=C(N,\omega,\epsilon)>0$ so: Let $0\in\Sigma^2\subset B_R\subset\mathbf{R}^3$ be an embedded minimal disk, $\partial\Sigma\subset\partial B_R$. If $\sup_{B_{r_0}\cap\Sigma}|A|^2\leq 4\,C^2\,r_0^{-2}$ and $|A|^2(0)=C^2\,r_0^{-2}$ for some $0<r_0<R$, then there exist $\bar{R}<r_0/\omega$ and (after a rotation) an N-valued graph $\Sigma_g\subset\Sigma$ over $D_{\omega\bar{R}}\setminus D_{\bar{R}}$ with gradient $\leq\epsilon$, and $\text{dist}_\Sigma(0,\Sigma_g)\leq 4\,\bar{R}$.

Recall that by the middle sheet Σ^M of an N-valued graph Σ we mean the portion over

$$\{(\rho,\theta)\in\mathcal{P}\,|\,r_1<\rho<r_2 \text{ and } 0\leq\theta\leq 2\,\pi\}\,. \tag{0.5}$$

The result that we need from [CM3] (combining theorem 0.3 and lemma II.3.8 there) is:

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Theorem 0.6. [CM3]. Given N_1 and $\tau > 0$, there exist $N, \Omega, \epsilon > 0$ so: If $\Omega\, r_0 < 1 < R_0/\Omega$, $\Sigma \subset B_{R_0}$ is an embedded minimal disk with $\partial \Sigma \subset \partial B_{R_0}$, and Σ contains an N-valued minimal graph Σ_g over $D_1 \setminus D_{r_0}$ with gradient $\leq \epsilon$ and $\Sigma_g \subset \{x_3^2 \leq \epsilon^2 (x_1^2 + x_2^2)\}$, then Σ contains a N_1-valued graph Σ_d over $D_{R_0/\Omega} \setminus D_{r_0}$ with gradient $\leq \tau$ and $(\Sigma_g)^M \subset \Sigma_d$.

As a consequence of Theorem 0.2, we will show that if $|A|^2$ is blowing up for a sequence of embedded minimal disks, then there is a smooth minimal graph through this point in the limit of a subsequence (Theorem 5.8 below).

Theorems 0.2, 0.4, 0.6, 5.8 are local and are for simplicity stated and proven only for $\mathbf{R}^3$ with the flat metric although they can with only very minor changes easily be seen to hold for a sufficiently small ball in any given fixed Riemannian 3-manifold.

Let x_1, x_2, x_3 be the standard coordinates on $\mathbf{R}^3$ and $\Pi : \mathbf{R}^3 \to \mathbf{R}^2$ orthogonal projection to $\{x_3 = 0\}$. For $y \in S \subset \Sigma \subset \mathbf{R}^3$ and $s > 0$, the extrinsic and intrinsic balls and tubes are

$$B_s(y) = \{x \in \mathbf{R}^3 \,|\, |x - y| < s\}\,, \quad T_s(S) = \{x \in \mathbf{R}^3 \,|\, \text{dist}_{\mathbf{R}^3}(x, S) < s\}\,, \tag{0.7}$$

$$\mathcal{B}_s(y) = \{x \in \Sigma \,|\, \text{dist}_\Sigma(x, y) < s\}\,, \mathcal{T}_s(S) = \{x \in \Sigma \,|\, \text{dist}_\Sigma(x, S) < s\}\,. \tag{0.8}$$

D_s denotes the disk $B_s(0) \cap \{x_3 = 0\}$. K_Σ the sectional curvature of a smooth compact surface Σ and when Σ is immersed A_Σ will be its second fundamental form. When Σ is oriented, $\mathbf{n}_\Sigma$ is the unit normal.

1. Poincaré and Caccioppoli type inequalities for area and curvature

In this section, we will first estimate the area of a surface (not necessarily minimal) in terms of its total curvature; see Corollary 1.5. This should be seen as analogous to a Poincaré inequality (for functions), and will be used similarly later in this paper. After that, we will bound the curvature by the area for a minimal disk; see Corollary 1.7. This inequality is similar to a Caccioppoli inequality and, unlike the Poincaré type inequality, relies on that the surface is minimal. Finally, we will apply these inequalities to show a strengthened (intrinsic) version of a result of Schoen and Simon.

Lemma 1.1. If $\mathcal{B}_{r_0}(x) \subset \Sigma^2$ is disjoint from the cut locus of x,

$$\text{Length}(\partial \mathcal{B}_{r_0}) - 2\pi r_0 = - \int_0^{r_0} \int_{\mathcal{B}_\rho} \text{K}_\Sigma\,, \tag{1.2}$$

$$\text{Area}(\mathcal{B}_{r_0}(x)) - \pi\, r_0^2 = - \int_0^{r_0} \int_0^{\tau} \int_{\mathcal{B}_\rho(x)} \text{K}_\Sigma\,. \tag{1.3}$$

Proof. For $0 < t \leq r_0$, by the Gauss-Bonnet theorem,

$$\frac{d}{dt} \int_{\partial \mathcal{B}_t} 1 = \int_{\partial \mathcal{B}_t} k_g = 2\,\pi - \int_{\mathcal{B}_t} \text{K}_\Sigma\,, \tag{1.4}$$

where k_g is the geodesic curvature of $\partial \mathcal{B}_t$. Integrating (1.4) gives the lemma. □

Corollary 1.5. If $\mathcal{B}_{r_0}(x) \subset \Sigma^2$ is disjoint from the cut locus of x,

$$\text{Area}(\mathcal{B}_{r_0}(x)) \leq \pi\, r_0^2 - \frac{1}{2}\, r_0^2 \int_{\mathcal{B}_{r_0}(x)} \min\{\text{K}_\Sigma, 0\}\ . \tag{1.6}$$

Corollary 1.7. If $\Sigma^2 \subset \mathbf{R}^3$ is immersed and minimal, $\mathcal{B}_{r_0} \subset \Sigma^2$ is a disk, and $\mathcal{B}_{r_0} \cap \partial \Sigma = \emptyset$,

$$t^2 \int_{\mathcal{B}_{r_0 - 2\,t}} |A|^2 \leq r_0^2 \int_{\mathcal{B}_{r_0}} |A|^2 \, (1 - r/r_0)^2/2 = \int_0^{r_0} \int_0^{\tau} \int_{\mathcal{B}_{\rho(x)}} |A|^2$$
$$= 2 \, (\text{Area}\,(\mathcal{B}_{r_0}) - \pi \, {r_0}^2) \leq r_0 \, \text{Length}(\partial \mathcal{B}_{r_0}) - 2\pi \, r_0^2 \, . \tag{1.8}$$

Proof. Since Σ is minimal, $|A|^2 = -2 \, \mathrm{K}_\Sigma$ and hence by Lemma 1.1

$$t^2 \int_{\mathcal{B}_{r_0 - 2\,t}} |A|^2 \leq t \int_0^{r_0 - t} \int_{\mathcal{B}_\rho} |A|^2 \leq \int_0^{r_0} \int_0^{\tau} \int_{\mathcal{B}_\rho} |A|^2 = 2 \, (\text{Area}\,(\mathcal{B}_{r_0}) - \pi \, {r_0}^2) \, . \tag{1.9}$$

The first equality follows by integration by parts twice (using the coarea formula). To get the last inequality in (1.8), note that $\frac{d^2}{dt^2}\text{Length}(\partial \mathcal{B}_t) \geq 0$ by (1.4) hence $\frac{d}{dt}\text{Length}(\partial \mathcal{B}_t) \geq t\,\text{Length}(\partial \mathcal{B}_t)$ and consequently $\frac{d}{dt}\,(\text{Length}(\partial \mathcal{B}_t)/t) \geq 0$. From this it follows easily. □

The following lemma and its corollary generalizes the main result of [ScSi]:

Lemma 1.10. Given C, there exists $\epsilon > 0$ so if $\mathcal{B}_{9s} \subset \Sigma \subset \mathbf{R}^3$ is an embedded minimal disk,

$$\int_{\mathcal{B}_{9s}} |A|^2 \leq C \text{ and } \int_{\mathcal{B}_{9s} \setminus \mathcal{B}_s} |A|^2 \leq \epsilon \, , \tag{1.11}$$

then $\sup_{\mathcal{B}_s} |A|^2 \leq s^{-2}$.

Proof. Observe first that for ϵ small, [CiSc] and (1.11) give

$$\sup_{\mathcal{B}_{8s} \setminus \mathcal{B}_{2s}} |A|^2 \leq C_1^2 \, \epsilon \, s^{-2} \, . \tag{1.12}$$

By (1.2) and (1.11)

$$\text{Length}(\partial \mathcal{B}_{2s}) \leq (4\pi + C) \, s \, . \tag{1.13}$$

We will next use (1.12) and (1.13) to show that, after rotating $\mathbf{R}^3$, $\mathcal{B}_{8s} \setminus \mathcal{B}_{2s}$ is (locally) a graph over $\{x_3 = 0\}$ and furthermore $|\Pi(\partial \mathcal{B}_{8\,s})| > 3\,s$. Combining these two facts with embeddedness, the lemma will then follow easily from Rado's theorem.

By (1.13), $\text{diam}(\mathcal{B}_{8s} \setminus \mathcal{B}_{2s}) \leq (12 + 2\pi + C/2)s$. Hence, integrating (1.12) gives

$$\sup_{x, x' \in \mathcal{B}_{8s} \setminus \mathcal{B}_{2s}} \text{dist}_{\mathbf{S}^2}(\mathbf{n}(x'), \mathbf{n}(x)) \leq C_1 \, \epsilon^{1/2} \, (12 + 2\pi + C/2) \, . \tag{1.14}$$

We can therefore rotate $\mathbf{R}^3$ so that

$$\sup_{\mathcal{B}_{8s} \setminus \mathcal{B}_{2s}} |\nabla x_3| \leq C_2 \, \epsilon^{1/2} \, (1 + C) \, . \tag{1.15}$$

Given $y \in \partial \mathcal{B}_{2s}$, let γ_y be the outward normal geodesic from y to $\partial \mathcal{B}_{8s}$ parametrized by arclength on $[0, 6s]$. Integrating (1.12) gives

$$\int_{\gamma_y|_{[0,t]}} |k_g^{\mathbf{R}^3}| \leq \int_{\gamma_y|_{[0,t]}} |A| \leq C_1 \, \epsilon^{1/2} \, t/s \, , \tag{1.16}$$

where $k_g^{\mathbf{R}^3}$ is the geodesic curvature of γ_y in $\mathbf{R}^3$. Combining (1.15) with (1.16) gives

$$\langle \nabla |\Pi(\cdot) - \Pi(y)|, \gamma_y' \rangle > 1 - C_3 \, \epsilon^{1/2} \, (1 + C) \, . \tag{1.17}$$

Integrating (1.17), we get that for ϵ small, $|\Pi(\partial \mathcal{B}_{8\,s})| > 3\,s$.

Combining $|\Pi(\partial \mathcal{B}_{8\,s})| > 3\,s$ and (1.15), it follows that, for ϵ small, $\Pi^{-1}(\partial D_{2s}) \cap \mathcal{B}_{8s}$ is a collection of immersed multi-valued graphs over ∂D_{2s}. Since $\mathcal{B}_{8s}$ is embedded, $\Pi^{-1}(\partial D_{2s}) \cap \mathcal{B}_{8s}$ consists of disjoint embedded circles which are graphs over ∂D_{2s}; this is the only use of embeddedness. Since $x_1^2 + x_2^2$ is subharmonic on the disk $\mathcal{B}_{8\,s}$, these circles bound disks in $\mathcal{B}_{8\,s}$ which are then graphs by Rado's theorem (see, e.g., [CM1]). The lemma now follows easily from (1.15) and the mean value inequality. □

Corollary 1.18. Given C_I, there exists C_P so if $\mathcal{B}_{2s} \subset \Sigma \subset \mathbf{R}^3$ is an embedded minimal disk with

$$\int_{\mathcal{B}_{2s}} |A|^2 \leq C_I \,, \tag{1.19}$$

then $\sup_{\mathcal{B}_s} |A|^2 \leq C_P\, s^{-2}$.

Proof. Let $\epsilon > 0$ be given by Lemma 1.10 with $C = C_I$ and then let N be the least integer greater than C_I/ϵ. Given $x \in \mathcal{B}_s$, there exists $1 \leq j \leq N$ with

$$\int_{\mathcal{B}_{9^{1-j}\,s}(x) \setminus \mathcal{B}_{9^{-j}\,s}(x)} |A|^2 \leq C_I/N \leq \epsilon \,. \tag{1.20}$$

Combining (1.19) and (1.20), Lemma 1.10 gives that $|A|^2(x) \leq (9^{-j} s)^{-2} \leq 9^{2N}\, s^{-2}$. □

We close this section with a generalization to surfaces of higher genus; see Theorem 1.22 below. This will not be used in this paper but will be useful in [CM6]. First we need:

Lemma 1.21. Let Σ be a surface and $\sigma \subset \Sigma$ a simple closed curve with length $\leq C\, r_0$. If for all $y \in \sigma$ the ball $\mathcal{B}_{r_0}(y)$ is a disk disjoint from $\partial\Sigma$, then there is a broken geodesic $\sigma_1 \subset \Sigma$ homotopic to σ in $\mathcal{T}_{r_0}(\sigma)$ and with $\leq C + 1$ breaks. If Σ is an annulus with $\mathrm{K}_\Sigma \leq 0$ and σ separates $\partial\Sigma$, then σ_1 contains a simple curve σ_2 homotopic to σ with $\leq C + 2$ breaks.

Proof. Parametrize σ by arclength so that $\sigma(0) = \sigma(\mathrm{Length}(\sigma))$. Let $0 = t_0 < \cdots < t_n = \mathrm{Length}(\sigma)$ be a subdivision with $t_{i+1} - t_i \leq r_0$ and $n \leq C + 1$. Since $\mathcal{B}_{r_0}(y)$ is a disk for all $y \in \sigma$, it follows that we can replace σ with a broken geodesic σ_1 with breaks at $\sigma(t_i) = \sigma(t_i)$ and which is homotopic to σ in $\mathcal{T}_{r_0}(\sigma)$.

Suppose also now that Σ is an annulus with $\mathrm{K}_\Sigma \leq 0$ and σ is topologically nontrivial. Let $[a, b]$ be a maximal interval so that $\sigma_1|_{[a,b]}$ is simple. We are done if $\sigma_1|_{[a,b]}$ is homotopic to σ. Otherwise, $\sigma_1|_{[a,b]}$ bounds a disk in Σ and the Gauss-Bonnet theorem implies that $\sigma_1|_{(a,b)}$ contains a break. Hence, replacing σ_1 by $\sigma_1 \setminus \sigma_1|_{(a,b)}$ gives a subcurve homotopic to σ but does not increase the number of breaks. Repeating this eventually gives σ_2. □

Given a surface Σ with boundary $\partial\Sigma$, we will define the *genus* of Σ ($\mathrm{gen}(\Sigma)$) to be the genus of the closed surface $\hat{\Sigma}$ obtained by adding a disk to each boundary circle. For example, the disk and the annulus are both genus zero; on the other hand, a closed surface of genus g with k disks removed has genus g.

In contrast to Corollary 1.18 (and the results preceding it), the next result concerns surfaces intersected with extrinsic balls. Below, $\Sigma_{0,s}$ is the component of $B_s \cap \Sigma$ with $0 \in \Sigma_{0,s}$.

Theorem 1.22. Given C_a, g, there exist C_c, C_r so: If $0 \in \Sigma \subset B_{r_0}$ is an embedded minimal surface with $\partial\Sigma \subset \partial B_{r_0}$, $\mathrm{gen}(\Sigma) \leq g$, $\mathrm{Area}(\Sigma) \leq C_a\, r_0^2$, and for each $C_r\, r_0 \leq s \leq r_0$, $\Sigma \setminus \Sigma_{0,s}$ is topologically an annulus, then Σ is a disk and $\sup_{\Sigma_{0,C_r\, r_0}} |A|^2 \leq C_c\, r_0^{-2}$.

Proof. By the coarea formula, we can find $r_0/2 \leq r_1 \leq 3r_0/4$ with $\text{Length}(\partial B_{r_1} \cap \Sigma) \leq 4\, C_a\, r_0$. It is easy to see from the maximum principle that $\mathcal{B}_{r_0/4}(y)$ is a disk for each $y \in \partial B_{r_1} \cap \Sigma$ (we will take $C_r < 1/4$). Applying Lemma 1.21 to $\partial \Sigma_{0,r_1} \subset \Sigma \setminus \Sigma_{0,r_0/4}$, we get a simple broken geodesic $\sigma_2 \subset \mathcal{T}_{r_0/4}(\partial \Sigma_{0,r_1})$ homotopic to $\partial \Sigma_{0,r_1}$ and with $\leq 16\, C_a + 2$ breaks. Consequently, the Gauss-Bonnet theorem gives

$$\int_{\Sigma_{0,r_0/4}} |A|^2 = -2 \int_{\Sigma_{0,r_0/4}} K_\Sigma \leq 8\,\pi\, g + 2 \int_{\sigma_2} |k_g| \leq 8\,\pi (g + 4\, C_a + 1)\,. \tag{1.23}$$

For $\epsilon > 0$, arguing as in Corollary 1.18 gives r_2 with $\int_{\Sigma_{0,5r_2} \setminus \Sigma_{0,r_2}} |A|^2 \leq \epsilon^2$ so, by [CiSc],

$$\sup_{\Sigma_{0,4r_2} \setminus \Sigma_{0,2r_2}} |A|^2 \leq C\, \epsilon^2\, r_2^{-2}\,. \tag{1.24}$$

Using the area bound, $\partial \Sigma_{0,3r_2}$ can be covered by $C C_a$ intrinsic balls $\mathcal{B}_{r_2/4}(x_i)$ with $x_i \in \partial \Sigma_{0,3r_2}$ (by the maximum principle, each $\mathcal{B}_{r_2/4}(x_i)$ is a disk). Hence, since $\partial \Sigma_{0,3r_2}$ is connected, any two points in $\partial \Sigma_{0,3r_2}$ can be joined by a curve in $\Sigma_{0,4r_2} \setminus \Sigma_{0,2r_2}$ of length $\leq C\, r_2$. Integrating (1.24) twice then gives a plane $P \subset \mathbf{R}^3$ with $\partial \Sigma_{0,3r_2} \subset T_{C\,\epsilon\, r_2}(P)$. By the convex hull property, $0 \in \Sigma_{0,3r_2} \subset T_{C\,\epsilon\, r_2}(P)$. Hence, since $\partial \Sigma_{0,3r_2}$ is connected and embedded, $\partial \Sigma_{0,3r_2}$ is a graph over the boundary of a convex domain for ϵ small. The standard existence theory and Rado's theorem give a minimal graph Σ_g with $\partial \Sigma_g = \partial \Sigma_{0,3r_2}$. By translating Σ_g above $\Sigma_{0,3r_2}$ and sliding it down to the first point of contact, and then repeating this from below, it follows easily from the strong maximum principle that $\Sigma_g = \Sigma_{0,3r_2}$, completing the proof. □

2. Finding large nearly stable pieces

We will collect here some results on stability of minimal surfaces which will be used later to conclude that certain sectors are nearly stable. The basic point is that two disjoint but nearby embedded minimal surfaces satisfying a priori curvature estimates must be nearly stable (made precise below). We start by recalling the definition of δ_s-stability. Let again $\Sigma \subset \mathbf{R}^3$ be an embedded oriented minimal surface.

Definition 2.1. (δ_s-stability). Given $\delta_s \geq 0$, set

$$L_{\delta_s} = \Delta + (1 - \delta_s)|A|^2\,, \tag{2.2}$$

so that L_0 is the usual Jacobi operator on Σ. A domain $\Omega \subset \Sigma$ is δ_s-stable if $\int \phi\, L_{\delta_s} \phi \leq 0$ for any compactly supported Lipschitz function ϕ (i.e., $\phi \in C_0^{0,1}(\Omega)$).

It follows that Ω is δ_s-stable if and only if, for all $\phi \in C_0^{0,1}(\Omega)$, we have the δ_s-stability inequality:

$$(1 - \delta_s) \int |A|^2 \phi^2 \leq \int |\nabla \phi|^2\,. \tag{2.3}$$

Since the Jacobi equation is the linearization of the minimal graph equation over Σ, standard calculations give:

Lemma 2.4. There exists $\delta_g > 0$ so that if Σ is minimal and u is a positive solution of the minimal graph equation over Σ (i.e., $\{x + u(x)\, \mathbf{n}_\Sigma(x)\,|\, x \in \Sigma\}$ is minimal) with $|\nabla u| + |u|\, |A| \leq \delta_g$, then $w = \log u$ satisfies on Σ

$$\Delta w = -|\nabla w|^2 + \text{div}(a \nabla w) + \langle \nabla w, a \nabla w \rangle + \langle b, \nabla w \rangle + (c - 1)|A|^2\,, \tag{2.5}$$

for functions a_{ij}, b_j, c on Σ with $|a|, |c| \leq 3\,|A|\,|u| + |\nabla u|$ and $|b| \leq 2\,|A|\,|\nabla u|$.

The following slight modification of a standard argument (see, e.g., proposition 1.26 of [CM1]) gives a useful sufficient condition for δ_s-stability of a domain:

Lemma 2.6. There exists $\delta > 0$ so: If Σ is minimal and $u > 0$ is a solution of the minimal graph equation over $\Omega \subset \Sigma$ with $|\nabla u| + |u|\,|A| \leq \delta$, then Ω is 1/2-stable.

Proof. Set $w = \log u$ and choose a cutoff function $\phi \in C_0^{0,1}(\Omega)$. Applying Stokes' theorem to $\operatorname{div}(\phi^2 \nabla w - \phi^2\, a\, \nabla w)$, substituting (2.5), and using $|a|, |c| \leq 3\,\delta, |b| \leq 2\,\delta\,|\nabla w|$ gives

$$(1 - 3\,\delta) \int \phi^2\,|A|^2 \leq - \int \phi^2\,|\nabla w|^2 + \int \phi^2\,\langle \nabla w, b + a\,\nabla w\rangle + 2 \int \phi \langle \nabla \phi, \nabla w - a\,\nabla w\rangle$$
$$\leq (5\delta - 1) \int \phi^2\,|\nabla w|^2 + 2(1 + 3\delta) \int |\phi\,\nabla w|\,|\nabla \phi|\,. \tag{2.7}$$

The lemma now follows easily from the absorbing inequality. □

We will use Lemma 2.6 to see that disjoint embedded minimal surfaces that are close are nearly stable (Corollary 2.13 below). Integrating $|\nabla \mathrm{dist}_{\mathbf{S}^2}(\mathbf{n}(x), \mathbf{n})| \leq |A|$ on geodesics gives

$$\sup_{x' \in \mathcal{B}_s(x)} \mathrm{dist}_{\mathbf{S}^2}(\mathbf{n}(x'), \mathbf{n}(x)) \leq s \sup_{\mathcal{B}_s(x)} |A|\,. \tag{2.8}$$

By (2.8), we can choose $0 < \rho_2 < 1/4$ so: If $\mathcal{B}_{2s}(x) \subset \Sigma$, $s \sup_{\mathcal{B}_{2s}(x)} |A| \leq 4\,\rho_2$, and $t \leq s$, then the component $\Sigma_{x,t}$ of $B_t(x) \cap \Sigma$ with $x \in \Sigma_{x,t}$ is a graph over $T_x\Sigma$ with gradient $\leq t/s$ and

$$\inf_{x' \in \mathcal{B}_{2s}(x)} |x' - x| / \mathrm{dist}_\Sigma(x, x') > 9/10\,. \tag{2.9}$$

One consequence is that if $t \leq s$ and we translate $T_x\Sigma$ so that $x \in T_x\Sigma$, then

$$\sup_{x' \in \mathcal{B}_t(x)} |x' - T_x\Sigma| \leq t^2/s\,. \tag{2.10}$$

Lemma 2.11. There exist $C_0, \rho_0 > 0$ so: If $\rho_1 \leq \min\{\rho_0, \rho_2\}$, $\Sigma_1, \Sigma_2 \subset \mathbf{R}^3$ are oriented minimal surfaces, $|A|^2 \leq 4$ on each Σ_i, $x \in \Sigma_1 \setminus \mathcal{T}_{4\rho_2}(\partial \Sigma_1)$, $y \in B_{\rho_1}(x) \cap \Sigma_2 \setminus \mathcal{T}_{4\rho_2}(\partial\Sigma_2)$, and $\mathcal{B}_{2\rho_1}(x) \cap \mathcal{B}_{2\rho_1}(y) = \emptyset$, then $\mathcal{B}_{\rho_2}(y)$ is the graph $\{z + u(z)\,\mathbf{n}(z)\}$ over a domain containing $\mathcal{B}_{\rho_2/2}(x)$ with $u \neq 0$ and $|\nabla u| + 4\,|u| \leq C_0\,\rho_1$.

Proof. Since $\rho_1 \leq \rho_2$, (2.9) implies that $\mathcal{B}_{2\rho_2}(x) \cap \mathcal{B}_{2\rho_2}(y) = \emptyset$. If $t \leq 9\rho_2/5$, then $|A|^2 \leq 4$ implies that the components $\Sigma_{x,t}, \Sigma_{y,t}$ of $B_t(x) \cap \Sigma_1, B_t(y) \cap \Sigma_2$, respectively, with $x \in \Sigma_{x,t}, y \in \Sigma_{y,t}$, are graphs with gradient $\leq t/(2\rho_2)$ over $T_x\Sigma_1, T_y\Sigma_2$ and have $\Sigma_{x,t} \subset \mathcal{B}_{2\rho_2}(x), \Sigma_{y,t} \subset \mathcal{B}_{2\rho_2}(y)$. The last conclusion implies that $\Sigma_{x,t} \cap \Sigma_{y,t} = \emptyset$. It now follows that $\Sigma_{x,t}, \Sigma_{y,t}$ are graphs over the same plane. Namely, if we set $\theta = \mathrm{dist}_{\mathbf{S}^2}(\mathbf{n}(x), \{\mathbf{n}(y), -\mathbf{n}(y)\})$, then (2.10), $|x - y| < \rho_1$, and $\Sigma_{x,t} \cap \Sigma_{y,t} = \emptyset$ imply that

$$\rho_1 - (t/2 - \rho_1)\,\sin\theta + t^2/(2\rho_2) > -t^2/(2\rho_2)\,. \tag{2.12}$$

Hence, $\sin\theta < \rho_1/(t/2 - \rho_1) + t^2/[(t/2 - \rho_1)\rho_2]$. For ρ_0/ρ_2 small, $\mathcal{B}_{\rho_2}(y)$ is a graph with bounded gradient over $T_x\Sigma_1$. The lemma now follows easily using the Harnack inequality. □

Combining Lemmas 2.6 and 2.11 gives:

Corollary 2.13. Given C_0, $\delta > 0$, there exists $\epsilon(C_0, \delta) > 0$ so that if $p_i \in \Sigma_i \subset \mathbf{R}^3$ $(i = 1, 2)$ are embedded minimal surfaces, $\Sigma_1 \cap \Sigma_2 = \emptyset$, $\mathcal{B}_{2R}(p_i) \cap \partial \Sigma_i = \emptyset$, $|p_1 - p_2| < \epsilon\, R$, and

$$\sup_{\mathcal{B}_{2R}(p_i)} |A|^2 \leq C_0\, R^{-2}\,, \tag{2.14}$$

then $\mathcal{B}_R(\tilde{p}_i) \subset \tilde{\Sigma}_i$ is δ-stable where $\tilde{p}_i$ is the point over p_i in the universal cover $\tilde{\Sigma}_i$ of Σ_i.

The next result gives a decomposition of an embedded minimal surface with bounded curvature into a portion with bounded area and a union of disjoint 1/2-stable domains.

Lemma 2.15. There exists C_1 so: If $0 \in \Sigma \subset B_{2R} \subset \mathbf{R}^3$ is an embedded minimal surface with $\partial \Sigma \subset \partial B_{2R}$, and $|A|^2 \leq 4$, then there exist disjoint 1/2-stable subdomains $\Omega_j \subset \Sigma$ and a function $\chi \leq 1$ which vanishes on $B_R \cap \Sigma \setminus \cup_j \Omega_j$ so that

$$\text{Area}(\{x \in B_R \cap \Sigma \,|\, \chi(x) < 1\}) \leq C_1\, R^3\,, \tag{2.16}$$

$$\int_{\mathcal{B}_R} |\nabla \chi|^2 \leq C_1\, R^3\,. \tag{2.17}$$

Proof. We can assume that $R > \rho_2$ (otherwise $B_R \cap \Sigma$ is stable). Let $\delta > 0$ be from Lemma 2.6 and C_0, ρ_0 be from Lemma 2.11. Set $\rho_1 = \min\{\rho_0/C_0, \delta/C_0, \rho_2/4\}$.

Given $x \in B_{2R-\rho_1} \cap \Sigma$, let Σ_x be the component of $B_{\rho_1}(x) \cap \Sigma$ with $x \in \Sigma_x$ and let B_x^+ be the component of $B_{\rho_1}(x) \setminus \Sigma_x$ which $\mathbf{n}(x)$ points into. Set

$$VB = \{x \in B_R \cap \Sigma \,|\, B_x^+ \cap \Sigma \setminus \mathcal{B}_{4\,\rho_1}(x) = \emptyset\} \tag{2.18}$$

and let $\{\Omega_j\}$ be the components of $B_R \cap \Sigma \setminus \overline{VB}$. Choose a maximal disjoint collection $\{\mathcal{B}_{\rho_1}(y_i)\}_{1 \leq i \leq \nu}$ of balls centered in VB. Hence, the union of the balls $\{\mathcal{B}_{2\,\rho_1}(y_i)\}_{1 \leq i \leq \nu}$ covers VB. Further, the "half-balls" $B_{\rho_1/2}(y_i) \cap B_{y_i}^+$ are pairwise disjoint. To see this, suppose that $|y_i - y_j| < \rho_1$ but $y_j \notin \mathcal{B}_{2\rho_1}(y_i)$. Then, by (2.9), $y_j \notin \mathcal{B}_{8\rho_1}(y_i)$ so $\mathcal{B}_{4\rho_1}(y_j) \notin B_{y_i}^+$ and $\mathcal{B}_{4\rho_1}(y_i) \notin B_{y_j}^+$; the triangle inequality then implies that $B_{\rho_1/2}(y_i) \cap B_{y_i}^+ \cap B_{\rho_1/2}(y_j) \cap B_{y_j}^+ = \emptyset$ as claimed. By (2.8)–(2.10), each $B_{\rho_1/2}(y_i) \cap B_{y_i}^+$ has volume approximately ρ_1^3 and is contained in B_{2R} so that $\nu \leq C\, R^3$. Define the function χ on Σ by

$$\chi(x) = \begin{cases} 0 & \text{if } x \in VB\,, \\ \text{dist}_\Sigma(x, VB)/\rho_1 & \text{if } x \in \mathcal{T}_{\rho_1}(VB) \setminus VB\,, \\ 1 & \text{otherwise}\;. \end{cases} \tag{2.19}$$

Since $\mathcal{T}_{\rho_1}(VB) \subset \cup_{i=1}^{\nu} \mathcal{B}_{3\,\rho_1}(y_i)$, $|A|^2 \leq 4$, and $\nu \leq C\, R^3$, we get (2.16). Combining (2.16) and $|\nabla \chi| \leq \rho_1^{-1}$ gives (2.17) (taking C_1 larger).

It remains to show that each Ω_j is 1/2-stable. Fix j. By construction, if $x \in \Omega_j$, then there exists $y_x \in B_x^+ \cap \Sigma \setminus \mathcal{B}_{4\,\rho_1}(x)$ minimizing $|x - y_x|$ in $B_x^+ \cap \Sigma$. In particular, by Lemma 2.11, $\mathcal{B}_{\rho_2}(y_x)$ is the graph $\{z + u_x(z)\,\mathbf{n}(z)\}$ over a domain containing $\mathcal{B}_{\rho_2/2}(x)$ with $u_x > 0$ and $|\nabla u_x| + 4\,|u_x| \leq \min\{\delta, \rho_0\}$. Choose a maximal disjoint collection of balls $\mathcal{B}_{\rho_2/6}(x_i)$ with $x_i \in \Omega_j$ and let $u_{x_i} > 0$ be the corresponding functions defined on $\mathcal{B}_{\rho_2/2}(x_i)$. Since Σ is embedded (and compact) and $|u_{x_i}| < \rho_0$, Lemma 2.11 implies that $u_{x_i}(x) = \min_{t>0}\{x + t\,\mathbf{n}(x) \in \Sigma\}$ for $x \in \mathcal{B}_{\rho_2/2}(x_i)$. Hence, $u_{x_{i_1}}(x) = u_{x_{i_2}}(x)$ for $x \in \mathcal{B}_{\rho_2/2}(x_{i_1}) \cap \mathcal{B}_{\rho_2/2}(x_{i_2})$. Note that $\mathcal{T}_{\rho_2/6}(\Omega_j) \subset \cup_i \mathcal{B}_{\rho_2/2}(x_i)$. We conclude that the u_{x_i}'s give a well-defined function $u_j > 0$ on $\mathcal{T}_{\rho_2/6}(\Omega_j)$ with $|\nabla u_j| + |u_j|\,|A| \leq \delta$. Finally, Lemma 2.6 implies that each Ω_j is 1/2-stable. $\square$

3. Total curvature and area of embedded minimal disks

Using the decomposition of Lemma 2.15, we next obtain polynomial bounds for the area and total curvature of intrinsic balls in embedded minimal disks with bounded curvature.

Lemma 3.1. There exists C_1 so if $0 \in \Sigma \subset B_{2R}$ is an embedded minimal disk, $\partial \Sigma \subset \partial B_{2R}$, $|A|^2 \leq 4$, then

$$\int_0^R \int_0^t \int_{\mathcal{B}_s} |A|^2 \, ds \, dt = 2(\text{Area}(\mathcal{B}_R) - \pi R^2) \leq 6 \pi R^2 + 20 \, C_1 \, R^5 \, . \tag{3.2}$$

Proof. Let C_1, χ, and $\cup_j \Omega_j$ be given by Lemma 2.15. Define ψ on $\mathcal{B}_R$ by $\psi = \psi(\text{dist}_\Sigma(0, \cdot)) = 1 - \text{dist}_\Sigma(0, \cdot)/R$, so $\chi \psi$ vanishes off of $\cup_j \Omega_j$. Using $\chi\psi$ in the 1/2-stability inequality, the absorbing inequality and (2.17) give

$$\begin{aligned} \int |A|^2 \chi^2 \psi^2 &\leq 2 \int |\nabla (\chi \psi)|^2 = 2 \int \left(\chi^2 |\nabla \psi|^2 + 2 \chi \, \psi \langle \nabla \chi, \nabla \psi \rangle + \psi^2 |\nabla \chi|^2 \right) \\ &\leq 6 \, C_1 \, R^3 + 3 \int \chi^2 |\nabla \psi|^2 \leq 6 \, C_1 \, R^3 + 3 \, R^{-2} \text{Area}(\mathcal{B}_R) \, . \end{aligned} \tag{3.3}$$

Using (2.16) and $|A|^2 \leq 4$, we get

$$\int |A|^2 \psi^2 \leq 4 \, C_1 \, R^3 + \int |A|^2 \chi^2 \psi^2 \leq 10 \, C_1 \, R^3 + 3 \, R^{-2} \, \text{Area} \, (\mathcal{B}_R) \, . \tag{3.4}$$

The lemma follows from (3.4) and Corollary 1.7. □

The polynomial growth allows us to find large intrinsic balls with a fixed doubling:

Corollary 3.5. There exists C_2 so that given $\beta, R_0 > 1$, we get R_2 so: If $0 \in \Sigma \subset B_{R_2}$ is an embedded minimal disk, $\partial \Sigma \subset \partial B_{R_2}$, $|A|^2(0) = 1$, and $|A|^2 \leq 4$, then there exists $R_0 < R < R_2/(2\,\beta)$ with

$$\int_{\mathcal{B}_{3R}} |A|^2 + \beta^{-10} \int_{\mathcal{B}_{2 \beta R}} |A|^2 \leq C_2 \, R^{-2} \, \text{Area} \, (\mathcal{B}_R) \, . \tag{3.6}$$

Proof. Set $\mathcal{A}(s) = \text{Area}(\mathcal{B}_s)$. Given m, Lemma 3.1 gives

$$\left(\min_{1 \leq n \leq m} \frac{\mathcal{A}((4\beta)^{2n} \, R_0)}{\mathcal{A}((4\beta)^{2n-2} \, R_0)} \right)^m \leq \frac{\mathcal{A}((4\beta)^{2m} \, R_0)}{\mathcal{A}(R_0)} \leq C_1' \, (4\beta)^{10m} \, R_0^3 \, . \tag{3.7}$$

Fix m with $C_1' \, R_0^3 < 2^m$ and set $R_2 = 2 \, (4\beta)^{2m} \, R_0$. By (3.7), there exists $R_1 = (4\beta)^{2n-2} \, R_0$ with $1 \leq n \leq m$ so

$$\frac{\mathcal{A}((4\beta)^2 \, R_1)}{\mathcal{A}(R_1)} \leq 2 \, (4\beta)^{10} \, . \tag{3.8}$$

For simplicity, assume that $\beta = 4^q$ for $q \in \mathbf{Z}^+$. As in (3.7), (3.8), we get $0 \leq j \leq q$ with

$$\frac{\mathcal{A}(4^{j+1} \, R_1)}{\mathcal{A}(4^j \, R_1)} \leq \left[\frac{\mathcal{A}(4\beta \, R_1)}{\mathcal{A}(R_1)} \right]^{1/(q+1)} \leq 2^{1/(q+1)} \, 4^{10} \, . \tag{3.9}$$

Set $R = 4^j \, R_1$. Combining (3.8), (3.9), and Corollary 1.7 gives (3.6). □

4. The local structure near the axis

Given $\gamma \subset \partial \mathcal{B}_r$, define the intrinsic sector

$$S_R(\gamma) = \{\exp_0(v) \,|\, r \leq |v| \leq r + R \text{ and } \exp_0(r\, v/|v|) \in \gamma\}\,. \tag{4.1}$$

The key for proving Theorem 0.4 is to find n large intrinsic sectors with a scale-invariant curvature bound. To do this, we first use Corollary 1.18 to bound $\text{Length}(\partial \mathcal{B}_R)/R$ from below for $R \geq R_0$. Corollary 3.5 gives $R_3 > R_0$ and n long disjoint curves $\tilde{\gamma}_i \subset \partial \mathcal{B}_{R_3}$ so the sectors over $\tilde{\gamma}_i$ have bounded $\int |A|^2$. Corollary 1.18 gives the curvature bound. Once we have these sectors, for n large, two must be close and hence, by Lemmas 2.6 and 2.11, 1/2-stable. The N-valued graph is then given by corollary II.1.34 of [CM3]:

Corollary 4.2. [CM3]. Given $\omega > 8, 1 > \epsilon > 0, C_0$, and N, there exist m_1, Ω_1 so: If $0 \in \Sigma$ is an embedded minimal disk, $\gamma \subset \partial \mathcal{B}_{r_1}$ is a curve, $\int_\gamma k_g < C_0\, m_1$, $\text{Length}(\gamma) = m_1\, r_1$, and $\mathcal{T}_{r_1/8}(S_{\Omega_1^2\, \omega\, r_1}(\gamma))$ is 1/2-stable, then (after rotating $\mathbf{R}^3$) $S_{\Omega_1^2\, \omega\, r_1}(\gamma)$ contains an N-valued graph Σ_N over $D_{\omega\, \Omega_1\, r_1} \setminus D_{\Omega_1\, r_1}$ with gradient $\leq \epsilon$, $|A| \leq \epsilon/r$, and $\text{dist}_{S_{\Omega_1^2\, \omega\, r_1}(\gamma)}(\gamma, \Sigma_N) < 4\, \Omega_1\, r_1$.

Proof. (of Theorem 0.4). Rescale by C/r_0 so that $|A|^2(0) = 1$ and $|A|^2 \leq 4$ on B_C.

Let C_2 be from Corollary 3.5 and then let $m_1, \Omega_1 > \pi$ be given by Corollary 4.2 with C_0 there $= 2\, C_2 + 2$. Fix a_0 large (to be chosen). By Corollaries 1.7, 1.18, there exists $R_0 = R_0(a_0)$ so that for any $R_3 \geq R_0$

$$a_0\, R_3 \leq R_3/4 \int_{\mathcal{B}_{R_3/2}} |A|^2 \leq \text{Length}(\partial \mathcal{B}_{R_3})\,. \tag{4.3}$$

Set $\beta = 2\Omega_1^2\, \omega$. Corollaries 1.7, 3.5 give $R_2 = R_2(R_0, \beta)$ so if $C \geq R_2$, then there is $R_0 < R_3 < R_2/(2\,\beta)$ with

$$\int_{\mathcal{B}_{3\,R_3}} |A|^2 + \beta^{-10} \int_{\mathcal{B}_{2\,\beta\,R_3}} |A|^2 \leq C_2\, {R_3}^{-2}\, \text{Area}(\mathcal{B}_{R_3}) \leq C_2\, \text{Length}(\partial \mathcal{B}_{R_3})/(2R_3)\,. \tag{4.4}$$

Using (4.3), choose n so that

$$a_0\, R_3 \leq 4\, m_1\, n\, R_3 < \text{Length}(\partial \mathcal{B}_{R_3}) \leq 8\, m_1\, n\, R_3\,, \tag{4.5}$$

and fix $2n$ disjoint curves $\tilde{\gamma}_i \subset \partial \mathcal{B}_{R_3}$ with length $2\, m_1\, R_3$. Define the intrinsic sectors

$$\tilde{S}_i = \{\exp_0(v) \,|\, 0 < |v| \leq 2\,\beta\, R_3 \text{ and } \exp_0(R_3\, v/|v|) \in \tilde{\gamma}_i\}\,. \tag{4.6}$$

Since the $\tilde{S}_i$'s are disjoint, combining (4.4) and (4.5) gives

$$\sum_{i=1}^{2n} \left(\int_{\mathcal{B}_{3\,R_3} \cap \tilde{S}_i} |A|^2 + \beta^{-10} \int_{\tilde{S}_i} |A|^2 \right) \leq 4\, C_2\, m_1\, n\,. \tag{4.7}$$

Hence, after reordering the $\tilde{\gamma}_i$, we can assume that for $1 \leq i \leq n$

$$\int_{\mathcal{B}_{3\,R_3} \cap \tilde{S}_i} |A|^2 + \beta^{-10} \int_{\tilde{S}_i} |A|^2 \leq 4\, C_2\, m_1\,. \tag{4.8}$$

Using the Riccati comparison theorem, there are curves $\gamma_i \subset \partial \mathcal{B}_{2R_3} \cap \tilde{S}_i$ with length $2\, m_1\, R_3$ so that if $y \in S_i = S_{\beta R_3}(\gamma_i) \subset \tilde{S}_i$, then $\mathcal{B}_{\text{dist}_\Sigma(0,y)/2}(y) \subset \tilde{S}_i$. Hence, by Corollary 1.18 and

(4.8), we get for $y \in S_i$ and $i \leq n$

$$\sup_{\mathcal{B}_{\text{dist}_\Sigma(0,y)/4}(y)} |A|^2 \leq C_3 \, \text{dist}_\Sigma^{-2}(0,y) \, , \tag{4.9}$$

where $C_3 = C_3(\beta, m_1)$. For $i \leq n$, (4.8) and the Gauss-Bonnet theorem yield

$$\int_{\gamma_i} k_g \leq 2\,\pi + 2\,C_2\,m_1 < (2\,C_2 + 2)\,m_1 \, . \tag{4.10}$$

By (4.9) and a Riccati comparison argument, there exists $C_4 = C_4(C_3)$ so that for $i \leq n$

$$1/(2R_3) \leq \min_{\gamma_i} k_g \leq \max_{\gamma_i} k_g \leq C_4/R_3 \, . \tag{4.11}$$

Applying Lemma 2.11 repeatedly (and using (4.9)), it is easy to see that there exists $\alpha > 0$ so that if $i_1 < i_2 \leq n$ and

$$\text{dist}_{C^1([0,2m_1],\mathbf{R}^3)}(\gamma_{i_1}/R_3, \gamma_{i_2}/R_3) \leq \alpha \, , \tag{4.12}$$

then $\{z + u(z)\,\mathbf{n}(z) \,|\, z \in \mathcal{T}_{R_3/4}(S_{i_1})\} \subset \cup_{y \in S_{i_2}} \mathcal{B}_{\text{dist}_\Sigma(0,y)/4}(y)$ for a function $u \neq 0$ with

$$|\nabla u| + |A|\,|u| \leq C_0' \, \text{dist}_{C^1([0,2m_1],\mathbf{R}^3)}(\gamma_{i_1}/R_3, \gamma_{i_2}/R_3) \ . \tag{4.13}$$

Here $\text{dist}_{C^1([0,2m_1],\mathbf{R}^3)}(\gamma_{i_1}/R_3, \gamma_{i_2}/R_3)$ is the scale-invariant C^1-distance between the curves.

Next, we use compactness to show that (4.12) must hold for n large. Namely, since each $\gamma_i/R_3 \subset B_2$ is parametrized by arclength on $[0, 2m_1]$ and has a uniform $C^{1,1}$ bound by (4.11), this set of maps is compact by the Arzela-Ascoli theorem. Hence, there exists n_0 so that if $n \geq n_0$, then (4.12) holds for some $i_1 < i_2 \leq n$. In particular, (4.13) and Lemma 2.6 imply that S_{i_1} is 1/2-stable for n large (now choose a_0, R_0, R_2). After rotating $\mathbf{R}^3$, Corollary 4.2 gives the N-valued graph $\Sigma_g \subset S_{i_1}$ over $D_{2\omega\,\Omega_1\,R_3} \setminus D_{2\Omega_1\,R_3}$ with gradient $\leq \epsilon$, $|A| \leq \epsilon/r$, and $\text{dist}_\Sigma(0, \Sigma_g) \leq 8\,\Omega_1\,R_3$. Rescaling by r_0/C, the theorem follows with $\bar{R} = 2\Omega_1\,R_3 r_0/C$. □

Corollary 4.14. Given $N > 1$ and $\tau > 0$, there exist $\Omega > 1$ and $C > 0$ so: Let $0 \in \Sigma^2 \subset B_R$ be an embedded minimal disk, $\partial\Sigma \subset \partial B_R$. If $R > r_0 > 0$ with $\sup_{B_{r_0} \cap \Sigma} |A|^2 \leq 4\,C^2\,r_0^{-2}$ and $|A|^2(0) = C^2\,r_0^{-2}$, then there exists (after a rotation) an N-valued graph $\Sigma_g \subset \Sigma$ over $D_{R/\Omega} \setminus D_{r_0}$ with gradient $\leq \tau$, $\text{dist}_\Sigma(0, \Sigma_g) \leq 4\,r_0$, and $\Sigma_g \subset \{x_3^2 \leq \tau^2\,(x_1^2 + x_2^2)\}$.

Proof. This follows immediately by combining Theorems 0.4 and 0.6. □

Proposition 4.15. There exists $\beta > 0$ so: If $\Sigma_g \subset \Sigma$ is as in Theorem 0.4, then the separation between the sheets of Σ_g over $\partial D_{\bar{R}}$ is at least $\beta\,\bar{R}$.

Proof. This follows easily from the curvature bound, Lemma 2.11, the Harnack inequality, and estimates for 1/2-stable surfaces. □

5. The blow up

Combining Corollary 4.14 and a blowup argument will give Theorem 0.2.

Lemma 5.1. If $0 \in \Sigma \subset B_{r_0}$, $\partial\Sigma \subset \partial B_{r_0}$, and $\sup_{B_{r_0/2} \cap \Sigma} |A|^2 \geq 16\,C^2\,r_0^{-2}$, then there exist $y \in \Sigma$ and $r_1 < r_0 - |y|$ with $|A|^2(y) = C^2\,r_1^{-2}$ and $\sup_{B_{r_1}(y) \cap \Sigma} |A|^2 \leq 4\,C^2\,r_1^{-2}$.

Proof. Set $F(x) = (r_0 - |x|)^2 \, |A|^2(x)$. Since $F \geq 0$, $F|\partial B_{r_0} \cap \Sigma = 0$, and Σ is compact, F achieves its maximum at $y \in \partial B_{r_0 - \sigma} \cap \Sigma$ with $0 < \sigma \leq r_0$. Since $\sup_{B_{r_0/2} \cap \Sigma} |A|^2 \geq 16 \, C^2 \, r_0^{-2}$,

$$F(y) = \sup_{B_{r_0} \cap \Sigma} F \geq 4 \, C^2 \, . \tag{5.2}$$

To get the first claim, define $r_1 > 0$ by

$$r_1^2 \, |A(y)|^2 = C^2 \, . \tag{5.3}$$

Since $F(y) = \sigma^2 \, |A(y)|^2 \geq 4 \, C^2$, we have $2 \, r_1 \leq \sigma$. Finally, by (5.2),

$$\sup_{B_{r_1}(y) \cap \Sigma} \left(\frac{\sigma}{2} \right)^2 |A|^2 \leq \sup_{B_{\frac{\sigma}{2}}(y) \cap \Sigma} \left(\frac{\sigma}{2} \right)^2 |A|^2 \leq \sup_{B_{\frac{\sigma}{2}}(y) \cap \Sigma} F \leq \sigma^2 \, |A(y)|^2 \, . \tag{5.4}$$

□

Proof. (of Theorem 0.2). This follows immediately from Corollary 4.14 and Lemma 5.1. □

If $y_i \in \Sigma_i$ is a sequence of minimal disks with $y_i \to y$ and $|A|(y_i)$ blowing up, then we can take $r_0 \to 0$ in Theorem 0.2. Combining this with the sublinear growth of the separation between the sheets from [CM3], we will get in Theorem 5.8 a smooth limit through y.

Below $\Sigma_{r,s}^{0,2\pi} \subset \Sigma$ is the "middle sheet" over $\{(\rho, \theta) \,|\, 0 \leq \theta \leq 2\pi, \; r \leq \rho \leq s\}$. The sublinear growth is given by proposition II.2.12 of [CM3]:

Proposition 5.5. [CM3]. Given $\alpha > 0$, there exist $\delta_p > 0, N_g > 5$ so: If Σ is a N_g-valued minimal graph over $D_{e^{N_g} R} \setminus D_{e^{-N_g} R}$ with gradient ≤ 1 and $0 < u < \delta_p \, R$ is a solution of the minimal graph equation over Σ with $|\nabla u| \leq 1$, then for $R \leq s \leq 2 \, R$

$$\sup_{\Sigma_{R,2R}^{0,2\pi}} |A_\Sigma| + \sup_{\Sigma_{R,2R}^{0,2\pi}} |\nabla u|/u \leq \alpha/(4 \, R) \, , \tag{5.6}$$

$$\sup_{\Sigma_{R,s}^{0,2\pi}} u \leq (s/R)^\alpha \sup_{\Sigma_{R,R}^{0,2\pi}} u \, . \tag{5.7}$$

Theorem 5.8. There exists $\Omega > 1$ so: Let $y_i \in \Sigma_i \subset B_R$ with $\partial \Sigma_i \subset \partial B_R$ be embedded minimal disks where $y_i \to 0$. If $|A_{\Sigma_i}|(y_i) \to \infty$, then, after a rotation and passing to a subsequence, there exist $\epsilon_i \to 0$, $\delta_i \to 0$, and 2-valued minimal graphs $\Sigma_{d,i} \subset \{x_3^2 \leq x_1^2 + x_2^2\} \cap \Sigma_i$ over $D_{R/\Omega} \setminus D_{\epsilon_i}$ with gradient ≤ 1, and separation at most $\delta_i \, s$ over ∂D_s. Finally, the $\Sigma_{d,i}$ converge (with multiplicity two) to a smooth minimal graph through 0.

Proof. The first part follows immediately from taking $r_0 \to 0$ in Theorem 0.2. When s is small, the bound on the separation follows from the gradient bound. The separation then grows less than linearly by Proposition 5.5, giving the bound for large s and showing that the $\Sigma_{d,i}$ close up in the limit. In particular, the $\Sigma_{d,i}$ converge to a minimal graph Σ' over $D_{R/\Omega} \setminus \{0\}$ with gradient ≤ 1 and $\Sigma' \subset \{x_3^2 \leq x_1^2 + x_2^2\}$. By a standard removable singularity theorem, $\Sigma' \cup \{0\}$ is a smooth minimal graph over $D_{R/\Omega}$. □

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