Preface

The motivation for these lecture notes on minimal surfaces is to cover the necessary background material needed for the papers [**?**], [**?**], [**?**], and [**?**] on compactness and convergence of minimal surfaces in three-manifolds. Some of these results are described in the last chapter of these notes. These results about convergence and compactness of embedded minimal surfaces in three-manifolds are in part motivated by a question of Pitts and Rubinstein. This question asks to give a bound for the Morse index of all embedded closed minimal surfaces of fixed genus in a closed three-manifold; see Chapter **??** for the precise statement. The claim of Pitts and Rubinstein is that if there is such a bound for a sufficiently large class of metrics on $\mathbf{S}^3$, then the famous spherical space-form problem can be settled affirmatively.

We also hope that these notes will help to stimulate interaction between minimal surface theory and the topology of three-manifolds.

These notes are an expanded version of a one-semester course taught at Courant in the spring of 1998. The only prerequisites needed are a basic knowledge of Riemannian geometry and some familiarity with the maximum principle. Of the various ways of approaching minimal surfaces (from complex analysis, PDE, or geometric measure theory), we have chosen to focus on the PDE aspects of the theory.

In Chapter 1, we will first derive the minimal surface equation as the Euler-Lagrange equation for the area functional on graphs. Subsequently, we derive the parametric form of the minimal surface equation (the first variation formula). The focus of the first chapter is on the basic properties of minimal surfaces, including the monotonicity formula for area and the Bernstein theorem. We also mention some examples. In the last section of Chapter 1, we derive the second variation formula, the stability inequality, and define the Morse index of a minimal surface.

Chapter **??** deals with generalizations of the Bernstein theorem discussed in Chapter 1. We begin the chapter by deriving Simons' inequality for the Laplacian of the norm squared of the second fundamental form of a minimal hypersurface Σ in $\mathbb{R}^n$. In the later sections, we discuss various applications of such an inequality. The first application that we give is to a theorem of Choi-Schoen giving curvature estimates for minimal surfaces with small total curvature. Using this estimate, we give a short proof of Heinz's curvature estimate for minimal graphs. Next, we discuss a priori estimates for stable minimal surfaces in three-manifolds, including estimates on area and total curvature of Colding-Minicozzi and the curvature estimate of Schoen. After that, we follow Schoen-Simon-Yau and combine Simons' inequality with the stability inequality to show higher L^p bounds for the square of the norm of the second fundamental form for stable minimal hypersurfaces. The higher L^p bounds are then used together with Simons' inequality to show curvature estimates for stable minimal hypersurfaces and to give a generalization due to De Giorgi, Almgren, and Simons of the Bernstein theorem proven in Chapter 1. We close the chapter with a discussion of minimal cones in Euclidean space and the relationship to the Bernstein theorem.

We start Chapter **??** by introducing stationary varifolds as a generalization of classical minimal surfaces. After that, we prove a generalization of the Bernstein theorem for minimal surfaces discussed in the preceding chapter. Namely, following [**?**], we will show in Chapter **??** that, in fact, a bound on the density gives an upper bound for the smallest affine subspace that the minimal surface lies in. We will deduce this theorem from the properties of the coordinate functions (in fact, more generally properties of harmonic functions) on k-rectifiable stationary varifolds of arbitrary codimension in Euclidean space.

Chapter **??** discusses the solution to the classical Plateau problem, focusing primarily on its regularity. The first three sections cover the basic existence result for minimal disks. After some general discussion of unique continuation and nodal sets, we study the local description of minimal surfaces in a neighborhood of either a branch point or a point of nontransverse intersection. Following Osserman and Gulliver, we rule out interior branch points for solutions of the Plateau problem. In the remainder of the chapter, we prove the embeddedness of the solution to the Plateau problem when the boundary is in the boundary of a mean convex domain. This last result is due to Meeks and Yau.

Finally, in Chapter **??**, we discuss the theory of minimal surfaces in three-manifolds. We begin by explaining how to extend the earlier results to this case (in particular, monotonicity, the strong maximum principle, and some of the other basic estimates for minimal surfaces). Next, we prove the compactness theorem of Choi and Schoen for embedded minimal surfaces in three-manifolds with positive Ricci curvature. An important point for this compactness result is that by results of Choi-Wang and Yang-Yau such minimal surfaces have uniform area bounds. The next section surveys recent results of [**?**], [**?**], [**?**], and [**?**] on compactness and convergence of minimal surfaces without area bounds. Finally, in the last section, we mention an application (from [**?**]) of the ideas of [**?**] to the study of complete minimal surfaces in $\mathbb{R}^3$.

Contents

CHAPTER 1

The Variation Formulas and Some Consequences

In this chapter, we will first derive the minimal surface equation as the Euler-Lagrange equation for the area functional on graphs. Subsequently, we derive the parametric form of the minimal surface equation (the first variation formula). The focus of the chapter is on some basic properties of minimal surfaces, including the monotonicity formula for area and the Bernstein theorem. We also mention some examples. In the last section, we derive the second variation formula, the stability inequality, and define the Morse index of a minimal surface.

1. The Minimal Surface Equation and Minimal Submanifolds

1.1. Graphs and the minimal surface equation. Suppose that $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function and consider the graph of the function u

$$\mathrm{Graph}_u = \{(x, y, u(x, y)) \mid (x, y) \in \Omega\}\,. \tag{1}$$

Then the area is

$$\begin{aligned}\mathrm{Area}(\mathrm{Graph}_u) &= \int_\Omega |(1, 0, u_x) \times (0, 1, u_y)| \\ &= \int_\Omega \sqrt{1 + u_x^2 + u_y^2} = \int_\Omega \sqrt{1 + |\nabla u|^2}\,,\end{aligned} \tag{2}$$

and the (upward pointing) unit normal is

$$N = \frac{(1, 0, u_x) \times (0, 1, u_y)}{|(1, 0, u_x) \times (0, 1, u_y)|} = \frac{(-u_x, -u_y, 1)}{\sqrt{1 + |\nabla u|^2}}\,. \tag{3}$$

Therefore for the graphs $\mathrm{Graph}_{u+t\eta}$ where $\eta|\partial\Omega = 0$ we get that

$$\mathrm{Area}(\mathrm{Graph}_{u+t\eta}) = \int_\Omega \sqrt{1 + |\nabla u + t\,\nabla \eta|^2} \tag{4}$$

hence

$$\begin{aligned} \frac{d}{dt}_{t=0} \text{Area}(\text{Graph}_{u+t\eta}) &= \int_\Omega \frac{\langle \nabla u \,, \nabla \eta \rangle}{\sqrt{1+|\nabla u|^2}} \\ &= -\int_\Omega \eta \text{ div } \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) . \end{aligned} \tag{5}$$

Therefore the graph of u is a critical point for the area functional if u satisfies the divergence form equation

$$\text{div} \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = 0 \,. \tag{6}$$

Equation (6) is the divergence form of the *minimal surface equation* and can alternatively be written as

$$\begin{aligned} 0 &= \left(1+|\nabla u|^2\right)^{\frac{3}{2}} \left[\left(\frac{u_x}{\sqrt{1+|\nabla u|^2}} \right)_x + \left(\frac{u_y}{\sqrt{1+|\nabla u|^2}} \right)_y \right] \\ &= (1+u_y^2)\, u_{xx} + (1+u_x^2)\, u_{yy} - 2\, u_x\, u_y\, u_{xy} \,. \end{aligned} \tag{7}$$

Next we want to show that the graph of a function on Ω satisfying the minimal surface equation is not just a critical point for the area functional but is actually area-minimizing amongst surfaces in the cylinder $\Omega \times \mathbb{R} \subset \mathbb{R}^3$. Let ω be the two-form on $\Omega \times \mathbb{R}$ given by that for X, $Y \in \mathbb{R}^3$

$$\omega(X,Y) = \text{det}(X,Y,N) \,, \tag{8}$$

where

$$N = \frac{(-u_x, -u_y, 1)}{\sqrt{1+|\nabla u|^2}} \,. \tag{9}$$

Observe that

$$\begin{aligned} \omega \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) &= \frac{1}{\sqrt{1+|\nabla u|^2}} \begin{vmatrix} 1 & 0 & -u_x \\ 0 & 1 & -u_y \\ 0 & 0 & 1 \end{vmatrix} \\ &= \frac{1}{\sqrt{1+|\nabla u|^2}} \,, \end{aligned} \tag{10}$$

$$\begin{aligned}\omega\left(\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right) &= \frac{1}{\sqrt{1+|\nabla u|^2}}\begin{vmatrix}0 & 0 & -u_x\\ 1 & 0 & -u_y\\ 0 & 1 & 1\end{vmatrix} \\ &= \frac{-u_x}{\sqrt{1+|\nabla u|^2}}\,,\end{aligned} \tag{11}$$

and

$$\begin{aligned}\omega\left(\frac{\partial}{\partial x},\frac{\partial}{\partial z}\right) &= \frac{1}{\sqrt{1+|\nabla u|^2}}\begin{vmatrix}1 & 0 & -u_x\\ 0 & 0 & -u_y\\ 0 & 1 & 1\end{vmatrix} \\ &= \frac{u_y}{\sqrt{1+|\nabla u|^2}}\,.\end{aligned} \tag{12}$$

Hence

$$\omega = \frac{dx\wedge dy - u_x\,dy\wedge dz - u_y\,dz\wedge dx}{\sqrt{1+|\nabla u|^2}} \tag{13}$$

and

$$d\omega = \frac{\partial}{\partial x}\left(\frac{-u_x}{\sqrt{1+|\nabla u|^2}}\right) + \frac{\partial}{\partial y}\left(\frac{-u_y}{\sqrt{1+|\nabla u|^2}}\right) = 0\,, \tag{14}$$

since u satisfies the minimal surface equation. In sum, the form ω is closed and, given any orthogonal unit vectors X and Y at a point (x, y, z),

$$|\omega(X,Y)| \leq 1\,, \tag{15}$$

where equality holds if and only if

$$X, Y \subset T_{(x,y,u(x,y))}\,\mathrm{Graph}_u\,. \tag{16}$$

Such a form ω is called a *calibration* and it can be used to show that Graph_u is area-minimizing:

LEMMA 1.1. *If* $u : \Omega \to \mathbb{R}$ *satisfies the minimal surface equation and* $\Sigma \subset \Omega \times \mathbb{R}$ *is any other surface with* $\partial\Sigma = \partial\ \mathrm{Graph}_u$*, then*

$$\mathrm{Area}(\mathrm{Graph}_u) \leq \mathrm{Area}(\Sigma)\,. \tag{17}$$

PROOF. Since ω is a closed form and Graph_u and Σ are homologous, Stokes' theorem gives

$$\int_{\mathrm{Graph}_u} \omega = \int_{\Sigma} \omega\,. \tag{18}$$

Combining this with (15) and (16) gives

$$\mathrm{Area}(\mathrm{Graph}_u) = \int_{\mathrm{Graph}_u} \omega = \int_{\Sigma} \omega \leq \mathrm{Area}(\Sigma)\,. \tag{19}$$

□

COROLLARY 1.2. *If $u : \Omega \to \mathbb{R}$ satisfies the minimal surface equation and $D_r \subset \Omega$, then*

$$\mathrm{Area}(B_r \cap \mathrm{Graph}_u) \leq \frac{\mathrm{Area}(\mathbf{S}^2)}{2}\, r^2 = 2\pi\, r^2\,. \tag{20}$$

PROOF. Since $\partial B_r \cap \mathrm{Graph}_u$ divides ∂B_r into two components at least one of which has area at most equal to $(\mathrm{Area}(\mathbf{S}^2)/2)\, r^2$, (17) gives (20). $\square$

If the domain Ω is convex, the minimal graph is absolutely area-minimizing. To see this, observe first that for a convex set Ω the nearest point projection $P : \mathbb{R}^3 \to \Omega \times \mathbb{R}$ is a distance nonincreasing Lipschitz map that is equal to the identity on $\Omega \times \mathbb{R}$. If $\Sigma \subset \mathbb{R}^3$ is any other surface with $\partial \Sigma = \partial\ \mathrm{Graph}_u$, then $\Sigma' = P(\Sigma)$ has $\mathrm{Area}(\Sigma') \leq \mathrm{Area}(\Sigma)$. Applying (17) to Σ', we see that $\mathrm{Area}(\mathrm{Graph}_u) \leq \mathrm{Area}(\Sigma')$ and the claim follows.

Very similar calculations to the ones above show that if $\Omega \subset \mathbb{R}^{n-1}$ and $u : \Omega \to \mathbb{R}$ is a C^2 function, then the graph of u is a critical point for the area functional if and only if u satisfies the equation

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0\,. \tag{21}$$

Moreover, as in (17), if u satisfies (21), then the graph of u is actually area-minimizing. Consequently, as in (20), if Ω contains a ball of radius r, then

$$\mathrm{Vol}(B_r \cap \mathrm{Graph}_u) \leq \frac{\mathrm{Vol}(\mathbf{S}^{n-1})}{2}\, r^{n-1}\,. \tag{22}$$

1.2. The geometry of submanifolds. We could also have looked more generally for a k-dimensional submanifold Σ possibly with boundary and sitting inside some Riemannian manifold M (with metric g and covariant derivative ∇) and which is a critical point for the area functional.

In the following, if X is a vector field on $\Sigma \subset M$, then we let X^T and X^N denote the tangential and normal components, respectively. The covariant derivative ∇ on M then induces a covariant derivative ∇_Σ on Σ and second fundamental form A of Σ. That is, the induced covariant derivative ∇_Σ is given by

$$\nabla_\Sigma = (\nabla)^T\,, \tag{23}$$

and the vector-valued bilinear form A on Σ is given for $X, Y \in T_x\Sigma$ by

$$A(X,Y) = (\nabla_X Y)^N\,. \tag{24}$$

Since the bracket of tangential vector fields is again a tangential vector field, it is easy to see that A is symmetric, i.e., $A(X,Y) = A(Y,X)$. Observe that

$$\begin{aligned} \sum_{\ell=1}^{n-k} g(A(X,Y), N_\ell)\, N_\ell &= \sum_{\ell=1}^{n-k} g(\nabla_X Y, N_\ell)\, N_\ell \\ &= -\sum_{\ell=1}^{n-k} g(Y, \nabla_X N_\ell)\, N_\ell \, , \end{aligned} \tag{25}$$

where N_ℓ is an orthonormal basis of vector fields for the normal space to Σ in a neighborhood of x.

The *mean curvature vector* H at x is by definition

$$H = \sum_{i=1}^{k} A(E_i, E_i)\, , \tag{26}$$

where E_i is an orthonormal basis for $T_x\Sigma$. Furthermore, the *norm squared of the second fundamental form* at x is by definition

$$|A|^2 = \sum_{i,j=1}^{k} |A(E_i, E_j)|^2 \, . \tag{27}$$

Recall also that the Gauss equations assert that if X, $Y \in T_x\Sigma$, then

$$\begin{aligned} &K_\Sigma(X,Y)\, |X \wedge Y|^2 = \\ &\quad K_M(X,Y)\, |X \wedge Y|^2 + g(A(X,X),\, A(Y,Y)) - g(A(X,Y),\, A(X,Y))\, , \end{aligned} \tag{28}$$

where

$$|X \wedge Y|^2 = g(X,X)\, g(Y,Y) - g(X,Y)^2 \tag{29}$$

and $K_M(X,Y)$ and $K_\Sigma(X,Y)$ are the sectional curvatures of M and Σ, respectively, in the two-plane spanned by the vectors X and Y. If $\Sigma^{n-1} \subset M^n$ is a hypersurface and N is a unit normal vector field in a neighborhood of x, then

$$\nabla_{(\cdot)} N : T_x\Sigma \to T_x\Sigma \tag{30}$$

is a symmetric map (often referred to as the Weingarten map) and its eigenvalues $(\kappa_i)_{i=1,\dots,n-1}$ are called the principal curvatures. Moreover,

$$g(H,N) = -\sum_{i=1}^{n-1} \kappa_i \, . \tag{31}$$

Finally, if X is a vector field defined in a neighborhood of Σ, then the *divergence* of X at $x \in \Sigma$ is

$$\operatorname{div}_\Sigma X = \sum_{i=1}^{n-1} g(\nabla_{E_i} X, E_i)\, , \tag{32}$$

where E_i is an orthonormal basis for $T_x\Sigma$. Notice that div_Σ satisfies the Leibniz rule

$$\mathrm{div}_\Sigma(fX) = \langle \nabla_\Sigma f, X\rangle + f\ \mathrm{div}_\Sigma(X)\,. \tag{33}$$

REMARK 1.3. *We can also use* div_Σ *to define the Laplace operator* Δ_Σ *on* Σ *by*

$$\Delta_\Sigma f = \mathrm{div}_\Sigma(\nabla_\Sigma f)\,. \tag{34}$$

REMARK 1.4. *Note that*

$$\begin{aligned}\mathrm{div}_\Sigma\, Y^N &= \sum_i g(E_i, \nabla_{E_i} Y^N) = -\sum_i g(Y^N, \nabla_{E_i} E_i)\\ &= -g(Y^N, H)\,.\end{aligned} \tag{35}$$

1.3. The first variation formula. Let $F : \Sigma \times (-\epsilon, \epsilon) \to M$ be a variation of Σ with compact support and fixed boundary. That is, $F = \mathrm{Id}$ outside a compact set,

$$F(x,0) = x\,, \tag{36}$$

and for all $x \in \partial\Sigma$

$$F(x,t) = x\,. \tag{37}$$

The vector field F_t restricted to Σ is often called the *variational vector field*. Now we want to compute the first variation of area for this one-parameter family of surfaces. Let x_i be local coordinates on Σ. Set

$$g_{ij}(t) = g(F_{x_i}, F_{x_j})\,, \tag{38}$$

$$\nu(t) = \sqrt{\det(g_{ij}(t))}\,\sqrt{\det(g^{ij}(0))}\,, \tag{39}$$

where a^{ij} denotes the inverse of the matrix a_{ij}. Note that $\nu(t)$ is well-defined independent of the choice of coordinate system on Σ (since $\det(g_{ij}(t))$ changes by the determinant squared of the differential of a coordinate transformation while $\det(g^{ij}(0))$ changes by the inverse of this). Furthermore, the area formula is

$$\mathrm{Vol}(F(\Sigma,t)) = \int \nu(t)\,\sqrt{\det(g_{ij}(0))}\,, \tag{40}$$

where the integral is over Σ. Differentiating this gives

$$\frac{d}{dt}_{t=0}\mathrm{Vol}(F(\Sigma,t)) = \int \frac{d}{dt}_{t=0}\nu(t)\,\sqrt{\det(g_{ij}(0))}\,. \tag{41}$$

To evaluate $d/dt_{t=0}\nu(t)$ at some point x, we may choose the coordinate system such that at x it is orthonormal, i.e., so that at the point x

$$g_{ij}(0) = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j\,,\\ 1 & \text{if } i = j\,.\end{cases} \tag{42}$$

Using this and the fact that the t and x_i derivatives commute (i.e., $\nabla_{F_t} F_{x_i} - \nabla_{F_{x_i}} F_t = [F_t, F_{x_i}] = 0$), we get at x,

$$\frac{d}{dt}_{t=0} \nu(t) = \frac{1}{2} \sum_{i=1}^{k} \frac{d}{dt} g(F_{x_i}, F_{x_i}) = \sum_{i=1}^{k} g(\nabla_{F_t} F_{x_i}, F_{x_i})$$
$$= \sum_{i=1}^{k} g(\nabla_{F_{x_i}} F_t, F_{x_i}) = \operatorname{div}_\Sigma F_t \, . \tag{43}$$

We can relate this formula to the mean curvature by writing the vector field F_t as the sum of its normal and tangential parts to get

$$\frac{d}{dt}_{t=0} \nu(t) = \sum_{\ell=1}^{n-k} \sum_{i=1}^{k} g(\nabla_{F_{x_i}} g(F_t, N_\ell)\, N_\ell, F_{x_i}) + \operatorname{div}_\Sigma \, F_t^T \tag{44}$$
$$= \sum_{\ell=1}^{n-k} \sum_{i=1}^{k} g(F_t, N_\ell)\, g(\nabla_{F_{x_i}} N_\ell, F_{x_i}) + \operatorname{div}_\Sigma \, F_t^T$$
$$= -g(F_t, H) + \operatorname{div}_\Sigma \, F_t^T \, .$$

Here N_ℓ is an orthonormal basis for the normal bundle of Σ at x. Integrating (43) and (44) gives the so-called first variation formula:

$$\frac{d}{dt}_{t=0} \operatorname{Vol}(F(\Sigma, t)) = -\int_\Sigma g(F_t, H) = \int_\Sigma \operatorname{div}_\Sigma F_t \, . \tag{45}$$

Note that Stokes' theorem was used to see that $\int \operatorname{div}_\Sigma \, F_t^T = 0$. As a consequence of (45), we see that Σ is a critical point for the area functional if and only if the mean curvature H vanishes identically.

DEFINITION 1.5. (Minimal Submanifold) An immersed submanifold $\Sigma^k \subset M^n$ is said to be *minimal* if the mean curvature H vanishes identically.

It follows from the identity (45) that a graph in $\mathbb{R}^3$ is a minimal surface if and only if it satisfies the minimal surface equation (6).

2. Some Simple Examples of Minimal Surfaces in $\mathbb{R}^3$

EXAMPLE 1.6. (A Plane) $x_3 = 0$. Planes are the only flat minimal surfaces.

EXAMPLE 1.7. (The Helicoid; see figure 1.) $x_3 = \tan^{-1}\left(\frac{x_2}{x_1}\right)$ which is given in parametric form by

$$(x_1, x_2, x_3) = (t \, \cos s, t \, \sin s, s) \, , \tag{46}$$

where s, $t \in \mathbb{R}$. This is a ruled surface since its intersections with horizontal planes $\{x_3 = s\}$ are straight lines. These lines lift and rotate with constant speed to form a double spiral staircase.

EXAMPLE 1.8. (The Catenoid; see figure 3.) $x_3 = \cosh^{-1} \sqrt{x_1^2 + x_2^2}$, that is, the surface obtained by rotating the curve $x_1 = \cosh x_3$ around the x_3-axis.

EXAMPLE 1.9. (Enneper's surface) The surface parameterized by

$$(x_1, x_2, x_3) = (s - s^3/3 + st^2, -t - s^2 t + t^3/3, s^2 - t^2)\,, \tag{47}$$

where s, $t \in \mathbb{R}$. Unlike the first three examples, Enneper's surface is not embedded.

EXAMPLE 1.10. (Scherk's Surface) Scherk's surface is the doubly–periodic surface given on the square $|x_1| < \pi/2$ and $|x_2| < \pi/2$ by

$$x_3 = \log \frac{\cos(x_2)}{\cos(x_1)}\,. \tag{48}$$

Let us check that Scherk's surface is, in fact, a minimal surface. We need only check that $x_3 = x_3(x_1, x_2)$ satisfies the minimal surface equation. Clearly

$$\partial_{x_1} x_3 = \tan(x_1) \tag{49}$$

$$\partial_{x_2} x_3 = -\tan(x_2) \tag{50}$$

$$\partial_{x_1 x_1} x_3 = 1 + \tan^2(x_1)\,, \tag{51}$$

$$\partial_{x_2 x_2} x_3 = -1 - \tan^2(x_2)\,, \tag{52}$$

$$\partial_{x_1 x_2} x_3 = 0\,. \tag{53}$$

Hence, x_3 satisfies the minimal surface equation (7).

FIGURE 1. Multi–valued graphs. The helicoid is obtained by gluing together two ∞–valued graphs along a line.

FIGURE 2. The catenoid given by revolving $x_1 = \cosh x_3$ around the x_3–axis.

3. Consequences of the First Variation Formula

In this section, we will collect some important consequences of the first variation formula. The most important of these is the monotonicity formula, Proposition 1.13. In later chapters, we will return to this subject. In Chapter 3, we extend these results to stationary varifolds, and in Chapter 5 to minimal surfaces in a three-manifold.

From (45), we see that Σ is minimal if and only if for all vector fields X with compact support and vanishing on the boundary of Σ,

$$\int_\Sigma \text{div}_\Sigma\, X = 0\,. \tag{54}$$

This equation is known as the first variation formula. It has the benefit that (54) makes sense as long as we can define the divergence on Σ. (This will later allow us to define a notion of "weak solution" for minimal surfaces.)

In the rest of this section, if $x_0 \in \mathbb{R}^n$ is fixed, then we let $B_s = B_s(x_0)$ be the Euclidean ball of radius s centered at x_0.

3.1. Harmonicity of the coordinate functions. As a consequence of (54), we will show the following proposition:

PROPOSITION 1.11. $\Sigma^k \subset \mathbb{R}^n$ *is minimal if and only if the restrictions of the coordinate functions of* $\mathbb{R}^n$ *to* Σ *are harmonic functions.*

PROOF. Let η be a smooth function on Σ with compact support and $\eta|\partial\Sigma = 0$, then

$$\int_\Sigma \langle \nabla_\Sigma \eta,\, \nabla_\Sigma x_i\rangle = \int_\Sigma \langle \nabla_\Sigma \eta\,, e_i\rangle = \int_\Sigma \text{div}_\Sigma(\eta\, e_i)\,. \tag{55}$$

From this, the claim follows easily. □

Another very useful consequence of (54) is a formula for the Laplacian of $|x|^2$ on a k-dimensional minimal surface Σ. Namely, since for any vector v we have

$$\nabla_v x_i = \langle v, e_i\rangle\,, \tag{56}$$

we see that

$$\text{div}_\Sigma(x_1, \ldots, x_n) = k\,. \tag{57}$$

Combining (57) and (54) (actually we just use that $\text{div}_\Sigma\, Y^N = 0$ for any Y) gives

$$\Delta_\Sigma |x|^2 = 2\ \text{div}_\Sigma(x_1, \ldots, x_n)^T = 2\ \text{div}_\Sigma(x_1, \ldots, x_n) = 2k\,. \tag{58}$$

Recall that if $\Xi \subset \mathbb{R}^n$ is a compact subset, then the smallest convex set containing Ξ (the convex hull, $\text{Conv}(\Xi)$) is the intersection of all half–spaces containing Ξ. The maximum principle forces a minimal submanifold to lie in the convex hull of its boundary:

PROPOSITION 1.12. (Convex Hull Property) *If* $\Sigma^k \subset \mathbb{R}^n$ *is a compact minimal submanifold, then* $\Sigma \subset \mathrm{Conv}(\partial\Sigma)$.

PROOF. A half–space $H \subset \mathbb{R}^n$ can be written as

$$H = \{x \in \mathbb{R}^n \,|\, \langle x, e\rangle \leq a\}\,, \tag{59}$$

for a vector $e \in \mathbf{S}^{n-1}$ and constant $a \in \mathbb{R}$. By Proposition 1.11, the function $u(x) = \langle e, x\rangle$ is harmonic on Σ and hence attains its maximum on $\partial\Sigma$ by the maximum principle. □

The argument in the proof of the convex hull property can be rephrased as saying that as we translate a hyperplane towards a minimal surface, the first point of contact must be on the boundary.

3.2. Monotonicity. Before we state and prove the monotonicity formula of volume for minimal submanifolds, we will need to recall the coarea formula. This formula asserts (see, for instance, [**?**] for a proof) that if Σ is a manifold and $h : \Sigma \to \mathbb{R}$ is a proper (i.e., $h^{-1}((-\infty, t])$ is compact for all $t \in \mathbb{R}$) Lipschitz function on Σ, then for all locally integrable functions f on Σ and $t \in \mathbb{R}$

$$\int_{\{h \leq t\}} f\,|\nabla_\Sigma h| = \int_{-\infty}^{t} \int_{h=\tau} f \, d\tau\,. \tag{60}$$

PROPOSITION 1.13. (The Monotonicity Formula) *Suppose that* $\Sigma^k \subset \mathbb{R}^n$ *is a minimal submanifold and* $x_0 \in \mathbb{R}^n$*; then for all* $0 < s < t$

$$t^{-k}\,\mathrm{Vol}(B_t \cap \Sigma) - s^{-k}\,\mathrm{Vol}(B_s \cap \Sigma) = \int_{(B_t \setminus B_s)\cap\Sigma} \frac{|(x-x_0)^N|^2}{|x-x_0|^{k+2}}\,. \tag{61}$$

PROOF. Since Σ is minimal,

$$\Delta_\Sigma |x - x_0|^2 = 2\,\mathrm{div}_\Sigma(x - x_0) = 2k\,. \tag{62}$$

By Stokes' theorem integrating this gives

$$2\,k\,\mathrm{Vol}(B_s \cap \Sigma) = \int_{B_s \cap \Sigma} \Delta_\Sigma |x-x_0|^2 = 2\int_{\partial B_s \cap \Sigma} |(x-x_0)^T|\,. \tag{63}$$

The coarea formula (i.e., (60)) gives

$$\mathrm{Vol}(B_s \cap \Sigma) = \int_{\{r \leq s\}} |\nabla_\Sigma r|^{-1}\,|\nabla_\Sigma r| = \int_0^s \int_{r=\tau} |\nabla_\Sigma r|^{-1}\, d\tau\,. \tag{64}$$

Combining this with (63), an easy calculation gives

$$\begin{aligned}\frac{d}{ds}\left(s^{-k}\,\mathrm{Vol}(B_s\cap\Sigma)\right) &= -k\,s^{-k-1}\,\mathrm{Vol}(B_s\cap\Sigma) + s^{-k}\int_{\partial B_s\cap\Sigma}\frac{|x-x_0|}{|(x-x_0)^T|}\\ &= s^{-k-1}\int_{\partial B_s\cap\Sigma}\left(\frac{|x-x_0|^2}{|(x-x_0)^T|} - |(x-x_0)^T|\right)\\ &= s^{-k-1}\int_{\partial B_s\cap\Sigma}\frac{|(x-x_0)^N|^2}{|(x-x_0)^T|}\,.\end{aligned} \tag{65}$$

Integrating and applying the coarea formula once more gives the claim. □

Notice that $(x-x_0)^N$ vanishes precisely when Σ is conical about x_0, i.e., when Σ is invariant under dilations about x_0. As a corollary, we get the following:

COROLLARY 1.14. *Suppose that* $\Sigma^k\subset\mathbb{R}^n$ *is a minimal submanifold and* $x_0\in\mathbb{R}^n$*; then the function*

$$\Theta_{x_0}(s)=\frac{\mathrm{Vol}(B_s\cap\Sigma)}{\mathrm{Vol}(B_s\subset\mathbb{R}^k)} \tag{66}$$

is a nondecreasing function of s*. Moreover,* $\Theta_{x_0}(s)$ *is constant in* s *if and only if* Σ *is conical about* x_0*.*

Finally, if $x_0\in\Sigma$*, then* $\Theta_{x_0}(s)\geq 1$*; if for some* $s>0$*,* $\Theta_{x_0}(s)=1$*, then* $B_s\cap\Sigma$ *is a ball in some* k*-dimensional plane.*

PROOF. Proposition 1.13 directly shows that $\Theta_{x_0}(s)$ is monotone nondecreasing. Since Σ is smooth and proper, it is infinitesimally Euclidean and hence

$$\lim_{s\to 0}\Theta_{x_0}(s)\geq 1\,. \tag{67}$$

Combining monotonicity of $\Theta_{x_0}(s)$ with (67) shows that $\Theta_{x_0}(s)\geq 1$. If we have $\Theta_{x_0}(s)=1$, then Θ_{x_0} is constant in s so that, by (61), $(x-x_0)^N$ is identically zero. Clearly this implies that Σ is dilation invariant, and since Σ is smooth, Σ is contained in a k-plane. □

For later reference, we will record some consequences of Corollary 1.14. Let Σ be a minimal submanifold and define the *density* at x_0 by

$$\Theta_{x_0}=\lim_{s\to 0}\Theta_{x_0}(s)\,. \tag{68}$$

This limit, which exists since $\Theta_{x_0}(s)$ is monotone, is always at least 1 for $x_0\in\Sigma$ by (67). In fact, so long as Σ is smooth, Θ_{x_0} is a nonnegative integer equal to the multiplicity of Σ at x_0. Note that if Σ is not embedded, then this multiplicity can be greater than one.

The next result, which is an elementary consequence of monotonicity, shows that this multiplicity is upper semicontinuous.

COROLLARY 1.15. *If $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold, then the density Θ_x is an upper semicontinuous function on $\mathbb{R}^n$. Consequently, for any $\Lambda \geq 0$, the set*

$$\{x \in \Sigma \mid \Theta_x \geq \Lambda\} \tag{69}$$

is closed.

PROOF. We need to show that if x_j is a sequence of points going to x, then

$$\Theta_x \geq \limsup_{x_j \to x} \Theta_{x_j} \, . \tag{70}$$

Given any $\delta > 0$, there exists an $s > 0$ such that

$$\Theta_x \geq \Theta_x(2s) - \delta \, , \tag{71}$$

and we can choose $0 < \epsilon < s$ so that

$$\Theta_x \geq (1 + s^{-1}\,\epsilon)^k \, \Theta_x(2s) - 2\,\delta \, . \tag{72}$$

For any x_j with $|x - x_j| < \epsilon$,

$$\begin{aligned} \Theta_{x_j} \leq \Theta_{x_j}(s) \leq \frac{\mathrm{Vol}(B_{s+\epsilon}(x) \cap \Sigma)}{\mathrm{Vol}(B_s \subset \mathbb{R}^k)} &= (1 + s^{-1}\,\epsilon)^k \, \Theta_x(s+\epsilon) \\ \leq 2\delta + \Theta_x \, , & \end{aligned} \tag{73}$$

where the last inequality follows from (72). Since δ was arbitrarily small, (73) implies (70) and hence Θ is upper semicontinuous. It follows immediately that the set defined in (69) must be closed. □

3.3. The meanvalue inequality.

PROPOSITION 1.16. (The Mean Value Inequality) *If $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold, $x_0 \in \mathbb{R}^n$, and f is a function on Σ, then*

$$t^{-k} \int_{B_t \cap \Sigma} f - s^{-k} \int_{B_s \cap \Sigma} f \tag{74}$$

$$= \int_{(B_t \setminus B_s) \cap \Sigma} f \, \frac{|(x - x_0)^N|^2}{|x - x_0|^{k+2}} + \frac{1}{2} \int_s^t \tau^{-k-1} \int_{B_\tau \cap \Sigma} (\tau^2 - |x - x_0|^2) \, \Delta_\Sigma f \, d\tau \, .$$

PROOF. Observe that the monotonicity formula will be the special case where $f = 1$. Since Σ is minimal, integration by parts gives

$$\begin{aligned} 2k \int_{B_s \cap \Sigma} f &= \int_{B_s \cap \Sigma} f \, \Delta_\Sigma |x - x_0|^2 \\ &= \int_{B_s \cap \Sigma} |x - x_0|^2 \, \Delta_\Sigma f + 2 \int_{\partial B_s \cap \Sigma} f \, |(x - x_0)^T| - s^2 \int_{B_s \cap \Sigma} \Delta_\Sigma f \, . \end{aligned} \tag{75}$$

Using this and the coarea formula (i.e., (60)) gives

$$\begin{aligned}
&\frac{d}{ds}\left(s^{-k} \int_{B_s \cap \Sigma} f \right) \qquad (76)\\
&\quad = -k\, s^{-k-1} \int_{B_s \cap \Sigma} f + s^{-k} \int_{\partial B_s \cap \Sigma} f \, \frac{|x - x_0|}{|(x-x_0)^T|} \\
&\quad = s^{-k-1} \int_{\partial B_s \cap \Sigma} f \, \frac{|(x-x_0)^N|^2}{|(x-x_0)^T|} + \frac{1}{2}\, s^{-k-1} \int_{B_s \cap \Sigma} (s^2 - |x - x_0|^2)\, \Delta_\Sigma f \, .
\end{aligned}$$

Integrating and using the coarea formula gives the claim. □

For future reference, we next record a general mean value inequality which follows from Proposition 1.16.

COROLLARY 1.17. *Suppose that* $\Sigma^k \subset \mathbb{R}^n$ *is a minimal submanifold,* $x_0 \in \Sigma$*, and* $s > 0$ *satisfy* $B_s(x_0) \cap \partial \Sigma = \emptyset$*. If* f *is a nonnegative function on* Σ *with* $\Delta_\Sigma f \geq -\lambda\, s^{-2}\, f$*, then*

$$f(x_0) \leq \mathrm{e}^{\frac{\lambda}{2}} \, \frac{\int_{B_s \cap \Sigma} f}{\mathrm{Vol}(B_s \subset \mathbb{R}^k)} \, . \tag{77}$$

PROOF. If we define $g(t)$ by

$$g(t) = t^{-k} \int_{B_t \cap \Sigma} f \, , \tag{78}$$

then Proposition 1.16 implies that

$$g'(t) \geq -\frac{\lambda}{2}\, s^{-2}\, t^{1-k} \int_{B_t \cap \Sigma} f = -\frac{\lambda}{2}\, s^{-2}\, t\, g(t) \, . \tag{79}$$

We can rewrite (79) as

$$\frac{g'(t)}{g(t)} \geq -\frac{\lambda}{2}\, s^{-2}\, t \geq -\frac{\lambda}{2s} \, . \tag{80}$$

From (80), it is obvious that $\mathrm{e}^{\lambda\, t/(2s)}\, g(t)$ is monotone nondecreasing and (77) follows immediately. □

We get immediately the following mean value inequality for the special case of nonnegative subharmonic functions:

COROLLARY 1.18. *Suppose that* $\Sigma^k \subset \mathbb{R}^n$ *is a minimal submanifold,* $x_0 \in \mathbb{R}^n$*, and* f *is a nonnegative subharmonic function on* Σ*; then*

$$s^{-k} \int_{B_s \cap \Sigma} f \tag{81}$$

is a nondecreasing function of s. In particular, if $x_0 \in \Sigma$, then for all $s > 0$

$$f(x_0) \leq \frac{\int_{B_s \cap \Sigma} f}{\mathrm{Vol}(B_s \subset \mathbb{R}^k)} \, . \tag{82}$$

4. The Gauss Map

Let $\Sigma^2 \subset \mathbb{R}^3$ be a surface. The *Gauss map* is a continuous choice of a unit normal

$$N : \Sigma \to \mathbf{S}^2 \subset \mathbb{R}^3 \, . \tag{83}$$

Observe that there are two choices of such a map N and $-N$ corresponding to a choice of orientation of Σ. Moreover, the differential of the map N can be identified with the Weingarten map defined above. To see this, suppose that E_1, E_2 is an orthonormal frame on Σ. Since the unit normal to $\mathbf{S}^2$ at $N(x)$ is just $N(x)$ itself, E_1, E_2 also gives an orthonormal frame on the image. Using this identification, the differential dN is given by

$$\langle dN(E_i), E_j \rangle = \langle \nabla_{E_i} N, E_j \rangle = -\langle N, \nabla_{E_i} E_j \rangle = -A_{ij} \, . \tag{84}$$

In the last equality, we identified the normal vector A_{ij} with its inner product with N (since Σ is a hypersurface).

If Σ is minimal, then the Gauss map is an (anti) conformal map since the eigenvalues of the Weingarten map are κ_1 and $\kappa_2 = -\kappa_1$. Moreover, for a minimal surface

$$|dN|^2 = |A|^2 = \kappa_1^2 + \kappa_2^2 = -2\,\kappa_1\,\kappa_2 = -2\,K = -2\det(dN) \, , \tag{85}$$

and the area of the Gauss map is a multiple of the total curvature. This conformality of the Gauss map for a minimal surface in $\mathbb{R}^3$, namely (85), can be used to prove the classical Bernstein theorem described in the next section.

4.1. Local coordinates on a graph.

We conclude this section by calculating the metric, curvature, and second fundamental form of a graph. If $\Sigma \subset \mathbb{R}^3$ is the graph of a function $u = u(x, y)$, then, as we have already seen, we can take

$$N = \frac{(-u_x, -u_y, 1)}{\sqrt{1 + |\nabla u|^2}} \, . \tag{86}$$

Using (x, y) as coordinates on the graph, we may express the induced metric g as

$$g_{xx} = (1 + u_x^2) \, , \quad g_{xy} = g_{yx} = u_x u_y \, , \quad g_{yy} = (1 + u_y^2) \, . \tag{87}$$

By direct calculation, the eigenvalues of the matrix g are 1 and $(1 + |\nabla u|^2)$. This can also easily be seen geometrically. Similarly, the inverse matrix is given by

$$g^{xx} = \frac{1+u_y^2}{1+|\nabla u|^2}\,, \quad g^{xy} = g^{yx} = \frac{-u_x\, u_y}{1+|\nabla u|^2}\,, \quad g^{yy} = \frac{1+u_x^2}{1+|\nabla u|^2}\,. \tag{88}$$

By the Gauss equation, the Gauss curvature of the graph of u is given by

$$\begin{aligned} K =& \kappa_1\, \kappa_2 = \frac{\langle N_x, (1,0,u_x)\rangle\, \langle N_y, (0,1,u_y)\rangle - \langle N_x, (0,1,u_y)\rangle^2}{|(1,0,u_x)\times(0,1,u_y)|^2} \\ =& \frac{\langle(-u_{xx},-u_{yx},0),(1,0,u_x)\rangle\,\langle(-u_{xy},-u_{yy},0),(0,1,u_y)\rangle}{(1+|\nabla u|^2)^2} \\ &- \frac{\langle(-u_{xx},-u_{xy},0),(0,1,u_y)\rangle^2}{(1+|\nabla u|^2)^2} = \frac{u_{xx}\,u_{yy}-u_{xy}^2}{(1+|\nabla u|^2)^2}\,. \end{aligned} \tag{89}$$

Therefore

$$K\, d\,\mathrm{Area} = \frac{u_{xx}\,u_{yy}-u_{xy}^2}{(1+|\nabla u|^2)^{\frac{3}{2}}}\; dx\wedge dy\,. \tag{90}$$

Similarly, we may express the second fundamental form A in the coordinates (x,y) as

$$\begin{aligned} A_{xx} &= \frac{u_{xx}}{(1+|\nabla u|^2)^{\frac{1}{2}}}\,, \\ A_{xy} = A_{yx} &= \frac{u_{xy}}{(1+|\nabla u|^2)^{\frac{1}{2}}}\,, \\ A_{yy} &= \frac{u_{yy}}{(1+|\nabla u|^2)^{\frac{1}{2}}}\,. \end{aligned} \tag{91}$$

Recall that (by (27)) the norm squared of A is given by

$$|A|^2 = A_{ij}\, A_{kl}\, g^{ik}\, g^{jl}\,. \tag{92}$$

The expression for the second fundamental form and the bound on the eigenvalues of g (see (87)) together imply

$$\frac{|\,\mathrm{Hess}_u\,|^2}{(1+|\nabla u|^2)^3} \leq |A|^2 \leq 2\frac{|\,\mathrm{Hess}_u\,|^2}{1+|\nabla u|^2}\,. \tag{93}$$

5. The Theorem of Bernstein

Before we prove the famous theorem of Bernstein, we will give a bound for the total curvature of a minimal graph. We will later see that, with some more work, this bound can be used to give local curvature estimates for minimal graphs. Such a local curvature estimate was proven originally by Heinz [**?**] using complex analysis and provided a generalization of the theorem of Bernstein.

LEMMA 1.19. *If $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a solution to the minimal surface equation, then for all nonnegative Lipschitz functions η with support contained in $\Omega \times \mathbb{R}$*

$$\int_{\mathrm{Graph}_u} |A|^2 \, \eta^2 \leq C \int_{\mathrm{Graph}_u} |\nabla_{\mathrm{Graph}_u} \eta|^2 \, . \tag{94}$$

PROOF. Let ω denote the area two-form on the unit sphere $\mathbf{S}^2$. Since the upper hemisphere is contractible, we can find a (smooth) one-form α on the upper hemisphere so that

$$d\alpha = \omega \, . \tag{95}$$

As before, let N denote the Gauss map. Since Σ is minimal and the differential d commutes with pull-backs, we see that

$$|A|^2 \, d\,\mathrm{Area} = -2K \, d\,\mathrm{Area} = 2\, N^* \omega = 2\, dN^* \alpha \, . \tag{96}$$

Moreover, since α is a one-form, there is a constant C_α so that

$$|N^* \alpha| \leq C_\alpha \, |dN| = C_\alpha \, |A| \, . \tag{97}$$

Set $\Sigma = \mathrm{Graph}_u$. By (96), Stokes' theorem, and (97), we get

$$\begin{aligned} \int_\Sigma \eta^2 \, |A|^2 \, d\,\mathrm{Area} = 2 \int_\Sigma \eta^2 \, dN^* \alpha &= -4 \int_\Sigma \eta \, d\eta \wedge N^* \alpha \\ &\leq 4\, C_\alpha \int_\Sigma \eta \, |\nabla_\Sigma \eta| \, |A| \, d\,\mathrm{Area} \\ &\leq 4\, C_\alpha \left(\int_\Sigma \eta^2 \, |A|^2 \, d\,\mathrm{Area} \right)^{\frac{1}{2}} \left(\int_\Sigma |\nabla_\Sigma \eta|^2 \, d\,\mathrm{Area} \right)^{\frac{1}{2}} , \end{aligned} \tag{98}$$

where the last inequality used the Cauchy-Schwarz inequality. Therefore

$$\int_\Sigma \eta^2 \, |A|^2 \leq 16\, C_\alpha^2 \int_\Sigma |\nabla_\Sigma \eta|^2 \, . \tag{99}$$

This proves the lemma. $\square$

COROLLARY 1.20. *If $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a solution to the minimal surface equation, $\kappa > 1$, and Ω contains a ball of radius $\kappa\, R$ centered at the origin, then*

$$\int_{B_{\sqrt{\kappa}\, R} \cap \mathrm{Graph}_u} |A|^2 \leq \frac{C}{\log \kappa} \, . \tag{100}$$

PROOF. Set $\Sigma = \mathrm{Graph}_u$. Define the cutoff function η on all of $\mathbb{R}^3$ and then restrict it to the graph of u as follows: Let r denote the distance to the origin in $\mathbb{R}^3$ and define η by

$$\eta = \begin{cases} 1 & \text{if } r^2 \leq \kappa\, R^2\,, \\ 2 - 2\,\log(r\,R^{-1})/\log\kappa & \text{if } \kappa\,R^2 < r^2 \leq \kappa^2\,R^2\,, \\ 0 & \text{if } r^2 > \kappa^2\,R^2\,. \end{cases} \tag{101}$$

Since $|\nabla_\Sigma r| \leq |\nabla r| = 1$, we have

$$|\nabla_\Sigma \eta| \leq \frac{2}{r\,\log\kappa}\,. \tag{102}$$

Applying Lemma 1.19 with this cutoff function η and using the area bound (20), we get

$$\begin{aligned} \int_{B_{\sqrt{\kappa}\,R}\cap\Sigma} |A|^2 &\leq \int_\Sigma \eta^2\,|A|^2 \leq C \int_\Sigma |\nabla_\Sigma \eta|^2 \leq \frac{4\,C}{(\log\kappa)^2} \int_{B_{\kappa\,R}\cap\Sigma} r^{-2} \\ &\leq \frac{4\,C}{(\log\kappa)^2} \sum_{\ell=(\log\kappa)/2}^{\log\kappa} \int_{(B_{\mathrm{e}^\ell\,R}\setminus B_{\mathrm{e}^{\ell-1}\,R})\cap\Sigma} r^{-2} \\ &\leq \frac{4\,C}{(\log\kappa)^2} \sum_{\ell=(\log\kappa)/2}^{\log\kappa} 2\,\pi\;\mathrm{e}^2 \leq \frac{4\,\pi\,C\;\mathrm{e}^2}{\log\kappa}\,. \end{aligned} \tag{103}$$

□

The above argument, i.e., integration by parts with this particular choice of η, is often referred to as “a logarithmic cutoff argument.” It is quite useful when the surface has at most quadratic area growth (as above).

As a consequence of this corollary, we get the following theorem of S. Bernstein [**?**] from 1916:

THEOREM 1.21. [**?**] *If* $u : \mathbb{R}^2 \to \mathbb{R}$ *is an entire solution to the minimal surface equation then* $u(x,y) = ax + by + c$ *for some constants* $a,\ b,\ c \in \mathbb{R}$.

PROOF. By the previous corollary, we have for all $R > 1$

$$\int_{B_{\sqrt{R}}\cap \mathrm{Graph}_u} |A|^2 \leq \frac{C}{\log R}\,. \tag{104}$$

Letting $R \to \infty$, we conclude that $|A|^2 \equiv 0$; hence $0 = u_{xx} = u_{xy} = u_{yy}$ and therefore $u = ax + by + c$ for some constants a, b, $c \in \mathbb{R}$. □

The previous proof (due to L. Simon [**?**]) of the theorem of Bernstein relied on minimality for two facts:

- The area bound for minimal graphs, (20).
- The conformality of the Gauss map, (85).

This proof can actually be applied to a wider class of differential equations where the conformality is replaced by quasi-conformality. We will briefly return to this later (in (**??**), where we also define quasi-conformality), but we will not discuss estimates for quasi-conformal maps in these notes. A detailed discussion may be found in chapter 16 of [**?**].

6. The Weierstrass representation

The classical Weierstrass representation takes holomorphic data (a Riemann surface, a meromorphic function, and a holomorphic one–form) and associates a minimal surface in $\mathbb{R}^3$. To be precise, given

- a Riemann surface Ω,
- a meromorphic function g on Ω,
- a holomorphic one–form ϕ on Ω,

then we get a (branched) conformal minimal immersion $F : \Omega \to \mathbb{R}^3$ by

$$F(z) = \text{Re} \int_{\zeta \in \gamma_{z_0,z}} \left(\frac{1}{2}\,(g^{-1}(\zeta) - g(\zeta)), \frac{i}{2}\,(g^{-1}(\zeta) + g(\zeta)), 1 \right) \, \phi(\zeta)\,. \tag{105}$$

Here $z_0 \in \Omega$ is a fixed base point and the integration is along a path $\gamma_{z_0,z}$ from z_0 to z. The choice of z_0 changes F by adding a constant. In general, the map F may depend on the choice of path (and hence may not be well–defined); this is known as "the period problem" (see M. Weber and M. Wolf, [**?**], for the latest developments).

LEMMA 1.22. *If f^1, f^2, f^3 are holomorphic functions on $\Omega \subset \mathbf{C}$ and $F = (F^1, F^2, F^3) : \Omega \to \mathbb{R}^3$ is given by*

$$F(z) = Re \int_{\zeta \in \gamma_{z_0,z}} \left(f^1, f^2, f^3\right) \, d\zeta\,, \tag{106}$$

then for each $i = 1, 2, 3$ we get

$$\frac{\partial F^i}{\partial x} - i\,\frac{\partial F^i}{\partial y} = f^i\,. \tag{107}$$

PROOF. If $g = u + i\,v$ is holomorphic, then the Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \tag{108}$$

$$\frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}\,. \tag{109}$$

In particular, we get

$$\begin{aligned} 2\,g' = \frac{\partial g}{\partial x} - i\,\frac{\partial g}{\partial y} &= \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - i\,\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \\ &= 2\,\left(\frac{\partial u}{\partial x} - i\,\frac{\partial u}{\partial y}\right)\,. \end{aligned} \tag{110}$$

□

As an immediate consequence of Lemma 1.22, we see that the Weierstrass representation gives a conformal mapping (see (112) below). Since harmonic functions are conformally invariant in dimension two, it follows that the restrictions of the coordinate functions to the image of a Weierstrass representation are harmonic. Consequently, we see that the image is in fact a minimal surface.

The unit normal N, metric ds^2, and Gauss curvature K of the resulting surface are then

$$N = \frac{(2\,\mathrm{Re}\,g, 2\,\mathrm{Im}\,g, |g|^2 - 1)}{|g|^2 + 1}\,, \tag{111}$$

$$ds^2 = \frac{|\phi|^2}{4}\,\left(|g|^{-1} + |g|\right)^2\,, \tag{112}$$

$$\mathrm{K} = -\left[\frac{4|\partial_z g|\,|g|}{|\phi|\,(1 + |g|^2)^2}\right]^2\,. \tag{113}$$

Since the pullback $F^*(dx_3)$ is Re ϕ by (105), ϕ is usually called the *height differential*. By (111), g is the composition of the Gauss map followed by stereographic projection.

The standard constructions of minimal surfaces from Weierstrass data are

$$g(z) = z,\ \phi(z) = dz/z,\ \Omega = \mathbf{C} \setminus \{0\} \text{ giving a catenoid}\,, \tag{114}$$

$$g(z) = \mathrm{e}^{iz},\ \phi(z) = dz,\ \Omega = \mathbf{C} \text{ giving a helicoid}\,, \tag{115}$$

$$g(z) = 1/z,\ \phi(z) = 2\,z\,dz,\ \Omega = \mathbf{C} \text{ giving Enneper's surface}\,. \tag{116}$$

The representation (105) gives an obvious way to deform a minimal immersion. Namely, multiplying the one–form ϕ by a unit complex number $\mathrm{e}^{i\theta}$ gives another minimal immersion. Moreover, since the metric ds^2 only depends on $|\phi|$, this new surface is isometric to the original one. When $\theta = \pi/2$, the new surface is called the *conjugate* minimal surface; for general values of θ, these are called associate minimal surfaces. Using a change of variables, it is not hard to see that the helicoid and the (universal cover of the) catenoid are conjugate minimal surfaces.

The Weierstrass representation is particularly useful for constructing immersed minimal surfaces. For example, in [**?**], Nadirashvili used it to construct a complete immersed minimal surface in the unit ball in $\mathbb{R}^3$. In particular, Nadirashvili's surface is not proper, i.e., the intersections with compact sets are not necessarily compact.

Typically, it is rather difficult to prove that the resulting immersion is an embedding (i.e., is 1–1), although there are some interesting cases where this can be done. The first modern example was [**?**] where D. Hoffman and W. Meeks proved that the surface constructed by Costa was embedded; this was the first new complete finite topology properly embedded minimal surface discovered since the classical catenoid, helicoid, and plane. This led to the discovery of many more such surfaces (see [**?**], [**?**], and [**?**] for more discussion).

7. The Strong Maximum Principle

First note that the difference of two solutions of the minimal surface equation satisfies a uniformly elliptic divergence form equation (where the bound on the ellipticity depends on the bounds for the gradients of the minimal graphs):

LEMMA 1.23. *If u_1 and u_2 are solutions of the minimal surface equation on a domain $\Omega \subset \mathbb{R}^n$, then $v = u_2 - u_1$ satisfies an equation of the form*

$$\operatorname{div}(a_{i,j}\, \nabla v) = 0\,, \tag{117}$$

where the eigenvalues of matrix $a_{i,j} = a_{i,j}(x)$ satisfy

$$0 < \mu \leq \lambda_1 \leq \cdots \leq \lambda_n \leq 1/\mu\,. \tag{118}$$

The constant μ depends only on the upper bounds for the gradients of $|\nabla u_i|$.

PROOF. Define the mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ by

$$F(X) = \frac{X}{(1+|X|^2)^{1/2}}\,. \tag{119}$$

By the fundamental theorem of calculus and the chain rule, we can write

$$\begin{aligned} F(\nabla u_2) - F(\nabla u_1) &= \int_0^1 \frac{d}{dt}\left(F(\nabla u_1 + t(\nabla u_2 - \nabla u_1))\right)\, dt \\ &= \int_0^1 dF(\nabla u_1 + t(\nabla u_2 - \nabla u_1))\, \nabla(u_2 - u_1)\, dt \\ &= \left(\int_0^1 dF(\nabla u_1 + t(\nabla u_2 - \nabla u_1))\, dt\right)\, \nabla(u_2 - u_1)\,. \end{aligned} \tag{120}$$

From this, we conclude that $v = u_2 - u_1$ satisfies an equation of the form

$$\operatorname{div}(a_{i,j}\, \nabla v) = 0\,, \tag{121}$$

where the matrix $a_{i,j}$ is given by (120).

Given a unit vector $V \in \mathbf{S}^{n-1}$ and $X \in \mathbb{R}^n$, we see that

$$dF(X)\,V = \frac{V}{(1+|X|^2)^{1/2}} - \frac{\langle X, V\rangle}{(1+|X|^2)^{3/2}}\,X\,. \tag{122}$$

In particular, taking the inner product with V gives

$$\begin{aligned}(1+|X|^2)^{3/2}\,\langle V, dF(X)\,V\rangle &= (1+|X|^2) - \langle X, V\rangle^2 \\ &\geq (1+|X|^2) - |X|^2 = 1\,. \end{aligned} \tag{123}$$

It follows that $a_{i,j}$ is a weighted average of positive definite matrices and thus also positive definite, completing the proof. □

The following corollary is the local version of the strong maximum principle for minimal hypersurfaces:

COROLLARY 1.24. *Let $\Omega \subset \mathbb{R}^n$ be an open connected neighborhood of the origin. If u_1, $u_2 : \Omega \to \mathbb{R}$ are solutions of the minimal surface equation with $u_1 \leq u_2$ and $u_1(0) = u_2(0)$, then $u_1 \equiv u_2$.*

PROOF. Since the matrix $a_{i,j}$ given by Lemma 1.23 is positive definite, we can apply the maximum principle for linear equations to $v = u_2 - u_1$ (see, for instance, [**?**] or theorem 3.5 of [**?**]). □

By writing a hypersurface locally as the graph of a function, we see that Corollary 1.24 has the following immediate consequence:

COROLLARY 1.25. (The Strong Maximum Principle) *If Σ_1, $\Sigma_2 \subset \mathbb{R}^n$ are complete connected minimal hypersurfaces (without boundaries), $\Sigma_1 \cap \Sigma_2 \neq \emptyset$, and Σ_2 lies on one side of Σ_1, then $\Sigma_1 = \Sigma_2$.*

As an application of the strong maximum principle, we next prove a generalization due to Schoen of a famous theorem of Rado (cf. Theorem **??**).

THEOREM 1.26. [**?**] *Let $\Omega \subset \mathbb{R}^2$ be strictly convex and $\sigma \subset \mathbb{R}^3$ a simple closed curve which is a graph over $\partial\Omega$ with bounded slope. Then any minimal surface $\Sigma \subset \mathbb{R}^3$ with $\partial\Sigma = \sigma$ must be graphical over Ω and hence unique.*

PROOF. We show that Σ is graphical and leave the uniqueness to the reader. By the maximum principle, the interior of Σ is contained in the interior of the cylinder $\Omega \times \mathbb{R}$. Given a plane $\{x_3 = t\}$, we divide Σ into the portions Σ_t^+ above and Σ_t^- below the plane. Reflecting Σ_t^+ below the plane gives a new minimal surface $\tilde{\Sigma}_t^+$ below the plane. By the maximum principle, there cannot be a first t where Σ_t^- and $\tilde{\Sigma}_t^+$ have an interior point of contact. Since $\partial\Sigma$ is a graph, there cannot be a boundary point of contact.

It follows then that the projection of Σ to the plane $\{x_3 = 0\}$ must be one to one, as desired. $\square$

The combination of the maximum principle and reflection used above is known as "the method of moving planes." It was originally developed by Alexandrov to prove that a closed embedded constant mean curvature hypersurface in $\mathbb{R}^n$ must be a round sphere.

8. Second Variation Formula, Morse Index, and Stability

8.1. The second variation formula. Suppose now that $\Sigma^k \subset M^n$ is a minimal submanifold; we want to compute the second derivative of the area functional for a variation of Σ. Therefore, let again F be a variation of Σ with compact support. In fact, we will assume that F is a normal variation, that is, on Σ we have

$$F_t^T \equiv 0\,. \tag{124}$$

As before, let x_i be local coordinates on Σ and set

$$g_{ij}(t) = g(F_{x_i}, F_{x_j})\,, \tag{125}$$

$$\nu(t) = \sqrt{\det(g_{ij}(t))}\,\sqrt{\det(g^{ij}(0))}\,. \tag{126}$$

Differentiating the measure $\nu(t)$ gives

$$\frac{d^2}{dt^2_{t=0}}\,\mathrm{Vol}(F(\Sigma,t)) = \int \frac{d^2}{dt^2}_{t=0}\nu(t)\,\sqrt{\det(g_{ij}(0))}\,. \tag{127}$$

Recall that the first derivative of the measure $\nu(t)$ can be written as

$$2\,\frac{d}{dt}\nu(t) = \mathrm{Tr}(g'_{ij}(t)g^{\ell m}(t))\,\nu(t)\,, \tag{128}$$

where the trace here means $\sum_{i,j} g'_{ij}(t)g^{ij}(t)$. To see this, recall that

$$\frac{d}{dt_{t=0}}\;\det(\delta_{ij} + t\,a_{ij}) = \mathrm{Tr}(a_{ij})\,. \tag{129}$$

To evaluate $d^2/dt^2_{t=0}\nu(t)$ at some point $x \in \Sigma$, we may choose the coordinate system x_i to be orthonormal at x. Differentiating (128) then gives at x

$$2\,\frac{d^2}{dt^2_{t=0}}\nu(t) = \mathrm{Tr}(g''_{ij}(0)) - \mathrm{Tr}(g'_{ij}(0)\,g'_{\ell m}(0)) + \frac{1}{2}\,[\mathrm{Tr}(g'_{ij}(0))]^2\,. \tag{130}$$

Since Σ is minimal, we have $\mathrm{Tr}(g'_{ij}(0)) = 0$ and, therefore, we get

$$2\,\frac{d^2}{dt^2_{t=0}}\nu(t) = \mathrm{Tr}(g''_{ij}(0)) - \mathrm{Tr}(g'_{ij}(0)\,g'_{\ell m}(0))\,. \tag{131}$$

An easy calculation gives

$$g'_{ij}(0) = g(F_{x_i t}, F_{x_j}) + g(F_{x_i}, F_{x_j t}) = -2\, g(A(F_{x_i}, F_{x_j}), F_t)\,, \tag{132}$$

and at x

$$\begin{aligned}
\sum_{i=1}^{k} g''_{ii}(0) &= 2\sum_{i=1}^{k} g(F_{x_i tt}, F_{x_i}) + 2\sum_{i=1}^{k} g(F_{x_i t}, F_{x_i t}) \\
&= 2\sum_{i=1}^{k} g(F_{x_i t}, F_{x_i t}) + 2\sum_{i=1}^{k} g(\mathrm{R}_M(F_t, F_{x_i})F_t, F_{x_i}) + 2\,\mathrm{div}_\Sigma(F_{tt}) \\
&= 2\sum_{i=1}^{k} g(F^T_{x_i t}, F^T_{x_i t}) + 2\sum_{i=1}^{k} g(F^N_{x_i t}, F^N_{x_i t}) \\
&\quad + 2\sum_{i=1}^{k} g(\mathrm{R}_M(F_t, F_{x_i})F_t, F_{x_i}) + 2\,\mathrm{div}_\Sigma(F_{tt}) \\
&= 2\sum_{i,j=1}^{k} |g(A(E_i, E_j), F_t)|^2 + 2|\nabla^N_\Sigma F_t|^2 \\
&\quad + 2\sum_{i=1}^{k} g(\mathrm{R}_M(F_t, E_i)F_t, E_i) + 2\,\mathrm{div}_\Sigma(F_{tt})\,.
\end{aligned} \tag{133}$$

Here R_M is the Riemann curvature tensor of M. To get the second equality, we used that at x

$$\begin{aligned}
\sum_{i=1}^{k} g(\nabla_{F_t}\nabla_{F_t}F_{x_i}, F_{x_i}) &= \sum_{i=1}^{k} g(\nabla_{F_t}\nabla_{F_{x_i}}F_t, F_{x_i}) \\
&= \sum_{i=1}^{k} g(\mathrm{R}_M(F_t, F_{x_i})F_t, F_{x_i}) + \sum_{i=1}^{k} g(\nabla_{F_{x_i}}\nabla_{F_t}F_t, F_{x_i}) \\
&= \sum_{i=1}^{k} g(\mathrm{R}_M(F_t, F_{x_i})F_t, F_{x_i}) + \mathrm{div}_\Sigma(F_{tt})\,.
\end{aligned} \tag{134}$$

Therefore, we get at x

$$\begin{aligned}
\frac{d^2}{dt^2_{t=0}}\nu(t) &= -\sum_{i,j=1}^{k} |g(F_t, A(E_i, E_j))|^2 \\
&\quad + |\nabla^N_\Sigma F_t|^2 - \sum_{i=1}^{k} g(\mathrm{R}_M(F_t, E_i)E_i, F_t) + \mathrm{div}_\Sigma(F_{tt})\,.
\end{aligned} \tag{135}$$

Inserting (135) into (127), integrating and using the minimality of Σ and Stokes' theorem, we get

$$\begin{aligned}\frac{d^2}{dt^2_{t=0}}\mathrm{Vol}(F(\Sigma,t)) &= -\sum_{i,j=1}^{k}\int_\Sigma |g(F_t, A(E_i,E_j))|^2 \\ &\quad + \int_\Sigma |\nabla^N_\Sigma F_t|^2 - \sum_{i=1}^{k}\int_\Sigma g(\mathrm{R}_M(F_t,E_i)E_i, F_t) \\ &= -\int_\Sigma g(F_t, L\,F_t)\,. \end{aligned} \tag{136}$$

The self-adjoint operator L is the so-called *stability operator* (or *Jacobi operator*) defined on a normal vector field X to Σ by

$$L\,X = \Delta^N_\Sigma\, X + \sum_{i=1}^{k}\mathrm{R}_M(X,E_i)\,E_i + \tilde{A}(X)\,, \tag{137}$$

where $\tilde{A}$ is *Simons' operator* defined by

$$\tilde{A}(X) = \sum_{i,j=1}^{k} g(A(E_i,E_j),X)\,A(E_i,E_j) \tag{138}$$

and Δ^N_Σ is the *Laplacian on the normal bundle*, that is,

$$\Delta^N_\Sigma\, X = \sum_{i=1}^{k}(\nabla_{E_i}\nabla_{E_i}X)^N - \sum_{i=1}^{k}(\nabla_{(\nabla_{E_i}E_i)^T}X)^N\,. \tag{139}$$

A normal vector field X with $L\,X = 0$ is said to be a *Jacobi field*.

For a hypersurfaces with a trivial normal bundle, the stability operator simplifies significantly since, in this case, it becomes an operator on functions. Namely, if we identify a normal vector field $X = \eta\,N$ with η, then

$$L\,\eta = \Delta_\Sigma\,\eta + |A|^2\,\eta + \mathrm{Ric}_M(N,N)\,\eta\,. \tag{140}$$

We will adopt the convention that λ is a (Dirichlet) eigenvalue of L on $\Omega \subset \Sigma$ if there exists a nontrivial normal vector field X which vanishes on $\partial\Omega$ so that

$$L\,X + \lambda X = 0\,. \tag{141}$$

DEFINITION 1.27. The *Morse index* of a compact minimal surface Σ is the number of negative eigenvalues of the stability operator L (counted with multiplicity) acting on the space of smooth sections of the normal bundle which vanish on the boundary.

The second variation formula shows that if $\Sigma^k \subset M^n$ is a minimal submanifold, then the Hessian of the area functional at Σ is given by

$$- \int_\Sigma g(\cdot, L\,\cdot)\,. \tag{142}$$

It follows that we could have equivalently defined the Morse index of Σ to be the index of Σ as a critical point for the area functional.

8.2. Stability. We say that a minimal submanifold $\Sigma^k \subset M^n$ is *stable* if for all variations F with boundary fixed

$$\frac{d^2}{dt^2_{t=0}} \operatorname{Vol}(F(\Sigma,t)) = - \int_\Sigma g(F_t, L\,F_t) \geq 0\,. \tag{143}$$

Observe that stability is the same as requiring the stability operator to be negative semidefinite (i.e., Morse index zero). Note also that if $\Sigma^{n-1} \subset \mathbb{R}^n$ is the graph of a function satisfying the minimal surface equation, then Σ is stable since Σ is, in fact, area-minimizing. A complete (possibly non-compact) minimal submanifold without boundary is said to be *stable* if all compact subdomains are stable.

For stable minimal hypersurfaces, we have the following useful inequality:

LEMMA 1.28. (The Stability Inequality) *Suppose that $\Sigma^{n-1} \subset M^n$ is a stable minimal hypersurface with trivial normal bundle, then for all Lipschitz functions η with compact support*

$$\int_\Sigma (\inf_M \operatorname{Ric}_M + |A|^2)\,\eta^2 \leq \int_\Sigma |\nabla_\Sigma \eta|^2\,. \tag{144}$$

PROOF. Since Σ is stable,

$$0 \leq - \int_\Sigma \eta\, L\eta = - \int_\Sigma (\eta\, \Delta_\Sigma\, \eta + |A|^2\, \eta^2 + \operatorname{Ric}_M(N,N)\, \eta^2)\,. \tag{145}$$

Integrating by parts gives

$$\int_\Sigma (\operatorname{Ric}_M(N,N) + |A|^2)\,\eta^2 \leq \int_\Sigma |\nabla_\Sigma \eta|^2\,. \tag{146}$$

This proves the lemma. □

The stability inequality gives restrictions on a stable minimal hypersurface when we have some positivity of the curvature of M. The next two corollary records two versions of this (the first due to J. Simons and the second to Schoen and Yau):

COROLLARY 1.29. *Suppose that $\Sigma^{n-1} \subset M^n$ is a closed stable minimal hypersurface with trivial normal bundle.*

- *If* $\mathrm{Ric}_M \geq 0$*, then* Σ *is totally geodesic and* $\mathrm{Ric}_M(N,N) = 0$ *on* Σ*.*
- *If* $\mathrm{Scal}_M > 0$ *and* $n = 3$*, then* Σ *is an* $\mathbf{S}^2$ *or an* $\mathbb{RP}^2$ *and*

$$\int_\Sigma (\mathrm{Scal}_M + |A|^2) \leq 8\,\pi\,. \tag{147}$$

PROOF. Since Σ is compact and has no boundary, we can use the constant function $\eta = 1$ in the stability inequality to get

$$\int_\Sigma (\mathrm{Ric}_M(N,N) + |A|^2) \leq 0\,. \tag{148}$$

The first conclusion follows immediately. For the second, let

$$E_1,\ E_2,\ \text{and } E_3 = N$$

be an orthonormal basis along Σ and use the Gauss equation to write

$$\begin{aligned} \mathrm{Ric}_M(N,N) &= R_{3113} + R_{3223} = 1/2\ \mathrm{Scal}_M - R_{2112} \\ &= 1/2\ \mathrm{Scal}_M - \mathrm{K}_\Sigma - 1/2\,|A|^2\,. \end{aligned} \tag{149}$$

Substituting this into (150), we get that

$$1/2 \int_\Sigma (\mathrm{Scal}_M + |A|^2) \leq - \int_\Sigma \mathrm{K}_\Sigma = -2\,\pi\,\chi(\Sigma)\,, \tag{150}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ and the equality used the Gauss-Bonnet formula. It follows that $\chi(\Sigma) > 0$, giving the claim. □

By using Lemma 1.28 and a logarithmic cutoff argument (as in the proof of Theorem 1.21), it is easy to give a second proof of Theorem 1.21. We will return to this point of view in the next chapter.

8.3. A characterization of stability. We will close this section with some useful characterizations of stability for minimal hypersurfaces with trivial normal bundle and we will derive some consequences. This will require more background in PDE than in the rest of these notes; when this occurs, precise references will be given.

For minimal hypersurfaces with trivial normal bundle, we saw that stability was equivalent to $\lambda_1(\Omega, L) \geq 0$ for every $\Omega \subset \Sigma$ where

$$\lambda_1(\Omega, L) = \inf \left\{ - \int \eta\, L\eta \mid \eta \in C_0^\infty(\Omega) \text{ and } \int_\Omega \eta^2 = 1 \right\}\,. \tag{151}$$

For smooth functions u, we define the H^1-norm by

$$\|u\|_{H^1}^2 = \int u^2 + \int |\nabla u|^2\,. \tag{152}$$

The Sobolev space $H_0^1(\Omega)$ is the closure of the compactly supported smooth functions on Ω with respect to the H^1-norm. Similarly, $H^1(\Omega)$ is the closure of the space of smooth functions on Ω with respect to the H^1-norm. By standard elliptic theory, see, for instance, [**?**] or [**?**], we get the following:

LEMMA 1.30. *Let* L *and* $\Omega \subset \Sigma$ *be as above and set* $\lambda_1 = \lambda_1(\Omega, L)$. *If* $u \in H_0^1(\Omega)$ *satisfies*

$$- \int_\Omega u\, Lu = \lambda_1 \int_\Omega u^2 \, , \tag{153}$$

then $Lu = -\lambda_1 u$.

PROOF. We can assume that

$$\int_\Omega u^2 = 1 \, . \tag{154}$$

If ψ is a smooth function with compact support in Ω and

$$\int \psi u = 0 \, , \tag{155}$$

then obviously

$$\frac{d}{dt}_{t=0} \int (u + t\psi)^2 = 0 \, . \tag{156}$$

By the definition of λ_1, (156) implies that

$$0 = \frac{d}{dt}_{t=0} \int (u + t\psi)\, L(u + t\psi) = 2 \int \psi\, Lu \, , \tag{157}$$

where the second equality follows from Stokes' theorem. By approximation, equation (157) holds for any $\psi \in H_0^1(\Omega)$ satisfying (155). In particular, given any $\phi \in H_0^1(\Omega)$, then (157) holds for

$$\psi = \phi - u \int (\phi u) \, , \tag{158}$$

and thus

$$\int \phi\, Lu = \int \phi u \int u\, Lu = -\lambda_1 \int \phi u \, . \tag{159}$$

Since (159) holds for all $\phi \in H_0^1(\Omega)$, u is a weak solution to $Lu = -\lambda_1 u$. The lemma now follows by elliptic regularity (theorem 8.8 of [**?**]). □

Combining Lemma 1.30 and the Harnack inequality, we see in the next lemma that any eigenfunction for the first eigenvalue cannot change sign.

LEMMA 1.31. *If* u *is a smooth function on* Ω *that vanishes on* $\partial\Omega$ *and* $Lu = -\lambda_1 u$ *where* $\lambda_1 = \lambda_1(\Omega, L)$, *then* u *cannot change sign in* Ω.

PROOF. We may assume that u is not identically zero. Since u vanishes on $\partial\Omega$, so does $|u|$. In fact, it is easy to see that $|u|$ also achieves the minimum in (151) and hence, by Lemma 1.30, we have $L\,|u| = -\lambda_1\,|u|$. Since $|u| \geq 0$ and $|u|$ is not identically zero, the Harnack inequality, Lemma 1.36, implies that $|u| > 0$ in Ω and the lemma follows. □

Since the eigenfunctions are all orthogonal to each other, Lemma 1.31 implies that only the lowest eigenfunction does not change sign and, in fact, the first eigenvalue has multiplicity one. As a consequence, we see that if $\Sigma \subset M$ is a stable minimal hypersurface with trivial normal bundle and without boundary, then $\tilde{\Sigma} \subset \tilde{M}$ is also stable where

$$G : \tilde{M} \to M \tag{160}$$

is a covering map, $\tilde{\Sigma} = G^{-1}(\Sigma)$, and $\tilde{M}$ is given the pullback metric. On the other hand, easy examples show that a cover of a stable minimal submanifold is not in general stable (consider, for instance, $\mathbb{RP}^2 \subset \mathbb{RP}^3$).

More generally, we have the following:

LEMMA 1.32. *Let Σ be a minimal hypersurface with trivial normal bundle, L its stability operator, and $\Omega \subset \Sigma$ a bounded domain. If there exists a positive function u on Ω with $Lu = 0$, then Ω is stable.*

PROOF. Set $q = |A|^2 + \mathrm{Ric}_M(N,N)$ so that $L = \Delta_\Sigma + q$. Since $u > 0$, $w = \log u$ is well-defined and satisfies

$$\Delta_\Sigma w = -q - |\nabla_\Sigma w|^2 \,. \tag{161}$$

Let f be a compactly supported smooth function on Ω. Multiplying both sides of (161) by f^2 and integrating by parts gives

$$\begin{aligned} \int f^2\, q + \int f^2\, |\nabla_\Sigma w|^2 = -\int f^2\, \Delta_\Sigma w \leq 2 \int |f|\, |\nabla_\Sigma f|\, |\nabla_\Sigma w| \\ \leq \int f^2\, |\nabla_\Sigma w|^2 + \int |\nabla_\Sigma f|^2 \,, \end{aligned} \tag{162}$$

where the second inequality follows from the Cauchy-Schwarz inequality. Cancelling the $\int f^2\, |\nabla_\Sigma w|^2$ term from both sides of (162), we see that

$$-\int f\, Lf \geq 0 \,. \tag{163}$$

Since this is true for any such f, the lemma follows. □

Note that if Σ is closed or, more generally, if u vanishes on $\partial\Sigma$, then Lemma 1.32 follows immediately from Lemma 1.31.

The variation $F : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ given by

$$F(\cdot, t) : (x_1, x_2, x_3) \to (x_1, x_2, x_3 + t) \tag{164}$$

is a one-parameter group of isometries of $\mathbb{R}^3$, and hence for any surface $\Sigma \subset \mathbb{R}^3$ we have that $\mathrm{Area}\; F(\Sigma, t)$ is constant. The variation vector field is $F_t = (0,0,1)$. If Σ is minimal, then the second variation formula implies that $\langle N, (0,0,1)\rangle$ (the normal component of the variation vector field) is a Jacobi field. Furthermore, when $\Sigma = \mathrm{Graph}_u$

$$\langle N, (0,0,1)\rangle = \frac{1}{\sqrt{1 + |\nabla_{\mathbb{R}^2} u|^2}} \, , \tag{165}$$

is therefore a positive Jacobi field. Consequently, Lemma 1.32 gives another way to see that minimal graphs are stable.

A manifold Σ is said to be *parabolic* if any positive superharmonic function u (i.e., $\Delta_\Sigma u \leq 0$) is constant. The next proposition uses the logarithmic cutoff trick to see that quadratic volume growth implies parabolicity. In this proposition, we will let $B_s^\Sigma = B_s^\Sigma(p)$ denote an intrinsic (geodesic) ball in Σ.

PROPOSITION 1.33. *If* Σ *is a complete surface so that for all* $s > 0$ *we have*

$$\mathrm{Area}\,(B_s^\Sigma) \leq C\, s^2 \, , \tag{166}$$

then Σ *is parabolic.*

PROOF. Suppose that $u > 0$ and $\Delta_\Sigma u \leq 0$ and set

$$w = \log u \, , \tag{167}$$

so that $|\nabla_\Sigma w|^2 \leq -\Delta_\Sigma w$. Let r denote the distance to p and define the cutoff function η by

$$\eta = \begin{cases} 1 & \text{if } r^2 \leq R \, , \\ 2 - \frac{\log r^2}{\log R} & \text{if } R < r^2 \leq R^2 \, , \\ 0 & \text{if } r^2 > R^2 \, . \end{cases} \tag{168}$$

By Stokes' theorem and the absorbing inequality ($2ab \leq \frac{1}{2}a^2 + 2b^2$), we get

$$\begin{aligned} \int \eta^2\, |\nabla_\Sigma w|^2 &\leq - \int \eta^2\, \Delta_\Sigma w \leq 2 \int \eta\, |\nabla_\Sigma \eta|\, |\nabla_\Sigma w| \\ &\leq \frac{1}{2} \int \eta^2\, |\nabla_\Sigma w|^2 + 2 \int |\nabla_\Sigma \eta|^2 \, . \end{aligned} \tag{169}$$

Substituting the definition of η and the area bound gives

$$\begin{aligned}\int_{B^{\Sigma}_{\sqrt{R}}} |\nabla_{\Sigma} w|^2 &\leq \int \eta^2 \, |\nabla_{\Sigma} w|^2 \leq 4 \int |\nabla_{\Sigma} \eta|^2 \\ &\leq \frac{16}{(\log R)^2} \sum_{\ell = \frac{1}{2}\log R}^{\log R} \int_{(B^{\Sigma}_{\mathrm{e}^{\ell}} \setminus B^{\Sigma}_{\mathrm{e}^{\ell-1}})} r^{-2} \qquad (170) \\ &\leq \frac{16}{(\log R)^2} \sum_{\ell = \frac{1}{2}\log R}^{\log R} C \, \mathrm{e}^2 \leq \frac{8\, C \; \mathrm{e}^2}{\log R} \, .\end{aligned}$$

Letting $R \to \infty$, we get that w is constant. □

Applying Proposition 1.33, we see that an entire two-dimensional minimal graph Σ is parabolic. This follows since the intrinsic distance is bounded from below by the Euclidean distance and therefore the area bound (20) implies that minimal graphs have quadratic area growth. Setting

$$u = \langle N, (0,0,1) \rangle \tag{171}$$

as in (165) gives a positive Jacobi field. In particular,

$$\Delta_{\Sigma} u = -(|A|^2 + \mathrm{Ric}_{\mathbb{R}^3}(N,N))\, u = -\, |A|^2 \, u \leq 0 \, , \tag{172}$$

so that u is a positive superharmonic function. By Proposition 1.33, u must be constant so that $\Delta_{\Sigma} u = 0$ and hence $|A|^2 = 0$. In other words, any complete minimal graph defined on $\mathbb{R}^2$ must be flat. This yields another proof of the Bernstein theorem, Theorem 1.21.

We will next give a characterization of stability for complete noncompact minimal hypersurfaces with trivial normal bundle due to Fischer-Colbrie and Schoen. We will assume that the boundary is smooth if it is nonempty.

PROPOSITION 1.34. [**?**] *If Σ is a complete noncompact minimal hypersurface with trivial normal bundle, then the following are equivalent:*

$$\lambda_1(\Omega, L) \geq 0 \textit{ for every bounded domain } \Omega \subset \Sigma \, . \tag{173}$$

$$\lambda_1(\Omega, L) > 0 \textit{ for every bounded domain } \Omega \subset \Sigma \, . \tag{174}$$

$$\textit{There exists a positive function } u \textit{ with } L\, u = 0 \, . \tag{175}$$

PROOF. By Lemma 1.32, (175) implies (173).

Clearly (174) implies (173). To see the equivalence of (173) and (174), given any bounded domain $\Omega_0 \subset \Sigma$ choose a strictly larger bounded domain Ω_1. The variational characterization of eigenvalues, (151), implies that

$$\lambda_1(\Omega_0, L) \geq \lambda_1(\Omega_1, L) \geq 0 \, , \tag{176}$$

where the second inequality follows from (173). Let u_0 denote the first eigenfunction for L on Ω_0, and define u_1 on Ω_1 by

$$u_1(x) = \begin{cases} u_0(x) & \text{if } x \in \Omega_0\,, \\ 0 & \text{otherwise}\ . \end{cases} \tag{177}$$

If we had equality in (176), then, by Lemma 1.30, $L\, u_1 = -\lambda_1 u_1$ on Ω_1 and, by Lemma 1.31, $u_1 > 0$ on Ω_1. This is not possible since u_1 vanishes on $\Omega_1 \setminus \Omega_0$, and thus the equivalence of (173) and (174) follows.

It remains to show that (174) implies (175). To do this, fix $p \in \Sigma$ and for each $r > 0$ let

$$B_r^\Sigma = B_r^\Sigma(p) = \{q \in \Sigma \mid \operatorname{dist}_\Sigma(p,q) < r\}\,. \tag{178}$$

Since $\lambda_1(B_r^\Sigma, L) > 0$, by the Fredholm alternative (see theorem 6.15 of [**?**]), there exists a unique function v_r with

$$L\, v_r = -|A|^2 - \operatorname{Ric}_M(N,N) \text{ on } B_r^\Sigma \quad \text{ and } \quad v_r = 0 \text{ on } \partial B_r^\Sigma\,. \tag{179}$$

Setting $u_r = v_r + 1$, (179) gives

$$L\, u_r = 0 \text{ on } B_r^\Sigma \quad \text{ and } \quad u_r = 1 \text{ on } \partial B_r^\Sigma\,. \tag{180}$$

We claim that:

$$u_r > 0 \text{ on } B_r^\Sigma\,. \tag{181}$$

By the Harnack inequality, Lemma 1.36, it suffices to show that $u_r \geq 0$ in B_r^Σ. If this fails, then we can choose a nonempty connected component Ω of the open set

$$\{x \in B_r^\Sigma \mid u_r(x) < 0\}\,. \tag{182}$$

By construction, we have $u_r < 0$ in Ω and $u_r = 0$ on $\partial\Omega$, and Lemma 1.31 implies that $\lambda_1(\Omega, L) = 0$. This contradiction implies that (181) holds.

For each r, we define a positive function w_r by

$$w_r = (u_r(p))^{-1}\, u_r \tag{183}$$

and observe that $Lw_r = 0$ and $w_r(p) = 1$.

Now, let K be any compact set with $K \subset B_{R_0}^\Sigma$. Applying the Harnack inequality (see theorem 8.27 of [**?**] for the estimates up to $\partial\Sigma$), we get for any $r \geq 2R_0$ that

$$\sup_K w_r \leq C_K\,. \tag{184}$$

The interior and boundary Schauder estimates (theorems 6.2 and 6.6 of [**?**]) imply that

$$|w_r|_{C_K^{2,\alpha}} \leq C'_K\,. \tag{185}$$

In other words, if $K \subset B_{R_0^\Sigma}$, we have uniform $C_K^{2,\alpha}$ estimates for every w_r for $r \geq 2R_0$. By the Arzela-Ascoli theorem, we can choose a subsequence of the w_r that converges uniformly in $C^{2,\frac{\alpha}{2}}$ on compact sets to a function

w. This convergence guarantees that w satisfies $Lw = 0$. Since each w_r was positive and $w_r(p) = 1$, w is nonnegative and has $w(p) = 1$. Finally, the Harnack inequality implies that w is also positive, which completes the proof. □

We can use Proposition 1.34 to give a slight generalization of the Bernstein theorem.

COROLLARY 1.35. *If $\Sigma \subset \mathbb{R}^3$ is a complete, connected, stable, parabolic, orientable minimal surface without boundary, then it must be a plane.*

PROOF. Since Σ is orientable and stable, Proposition 1.34 implies that there exists a function $u > 0$ with

$$\Delta_\Sigma u = -(|A|^2 + \mathrm{Ric}_{\mathbb{R}^3}(N,N))\, u = -\,|A|^2\, u \leq 0\,. \qquad (186)$$

Since Σ is parabolic, u must be constant. Hence (186) implies that $|A| \equiv 0$ and the corollary follows. □

9. Basic properties of multi–valued graphs

We will need the notion of a multi–valued graph, each staircase will be a multi–valued graph. Intuitively, an (embedded) multi–valued graph is a surface such that over each point of the annulus, the surface consists of N graphs. To make this notion precise, let D_r be the disk in the plane centered at the origin and of radius r and let $\mathcal{P}$ be the universal cover of the punctured plane $\mathbf{C} \setminus \{0\}$ with global polar coordinates (ρ, θ) so $\rho > 0$ and $\theta \in \mathbb{R}$. An *N–valued graph* on the annulus $D_s \setminus D_r$ is a single valued graph of a function u over

$$\{(\rho,\theta)\,|\, r < \rho \leq s\,,\ |\theta| \leq N\,\pi\}\,. \qquad (187)$$

For working purposes, we generally think of the intuitive picture of a multi–sheeted surface in $\mathbb{R}^3$, and we identify the single–valued graph over the universal cover with its multi–valued image in $\mathbb{R}^3$.

The multi–valued graphs that we will consider will all be embedded, which corresponds to a nonvanishing separation between the sheets (or the floors). Here the *separation* is the function (see fig. 3)

$$w(\rho,\theta) = u(\rho,\theta + 2\pi) - u(\rho,\theta)\,. \qquad (188)$$

If Σ is the helicoid, then $\Sigma \setminus \{x_3 - \text{axis}\} = \Sigma_1 \cup \Sigma_2$, where Σ_1, Σ_2 are ∞–valued graphs on $\mathbf{C} \setminus \{0\}$. Σ_1 is the graph of the function $u_1(\rho,\theta) = \theta$ and Σ_2 is the graph of the function $u_2(\rho,\theta) = \theta + \pi$. ($\Sigma_1$ is the subset where $s > 0$ in (46) and Σ_2 the subset where $s < 0$.) In either case the separation $w = 2\,\pi$. A *multi–valued minimal graph* is a multi–valued graph of a function u satisfying the minimal surface equation.

Note that for an embedded multi–valued graph, the sign of w determines whether the multi–valued graph spirals in a left–handed or right–handed manner, in other words, whether upwards motion corresponds to turning in a clockwise direction or in a counterclockwise direction.

FIGURE 3. The separation w for a multi–valued minimal graph; see (188).

It is easy to see that a multi-valued graph must be stable. Namely, the function $\langle N, (0,0,1)\rangle$ is a non-vanishing Jacobi field so stability follows immediately from Lemma 1.32.

Finally, we note that the separation w satisfies a reasonably nice divergence form equation. Namely, Lemma 1.23 implies that

$$\operatorname{div}(a_{i,j}\, \nabla w) = 0\,, \tag{189}$$

where the matrix $a_{i,j}$ is bounded and positive definite. In fact, the eigenvalues of $a_{i,j}$ are uniformly bounded in terms of the maximum of the gradient of the multi-valued function u. Note that this is enough to get a Harnack inequality for w so long as the gradient of u is uniformly bounded.

10. Exercises

EX 1. *Prove that the area functional on graphs is convex.*

EX 2. *Prove that the catenoid and helicoid are minimal surfaces.*

EX 3. *Verify that the Weierstrass representation*

$$F(z) = \textit{Re} \int_{\zeta \in \gamma_{z_0, z}} \left(\frac{1}{2}\,(g^{-1}(\zeta) - g(\zeta)), \frac{i}{2}\,(g^{-1}(\zeta) + g(\zeta)), 1 \right) \phi(\zeta)\,,$$

where $g(z) = z$, $\phi(z) = dz/z$, $\Omega = \mathbf{C} \setminus \{0\}$ *gives a catenoid.*

EX 4. *Derive the formulas* (111)–(113) *for the unit normal, metric, and curvature of a minimal surface from its Weierstrass data.*

EX 5. *Compute the total curvature, i.e.,* $\int K$, *of the catenoid.*

EX 6. *Find all rotationally symmetric solutions of the minimal surface equation on* $\mathbb{R}^2 \setminus D_1$.

EX 7. *Prove that Enneper's surface is not embedded but its unit normal is asymptotically vertical (i.e., like a graph). What does Enneper's surface look like asymptotically?*

Ex 8. *Suppose that M is a three-manifold with sectional curvatures at most -1.*

- *Prove that there is no minimal immersion of $\mathbf{S}^2$ into M.*
- *Prove an area bound for a minimal immersion of a surface of genus g into M.*

Ex 9. *Suppose that $u : D_1 \to \mathbb{R}$ satisfies the minimal surface equation and $|Du| \leq 1$ on D_1. Use standard elliptic theory to prove a uniform bound $|D^2 u|(0) \leq C$.*

Ex 10. *Find a sequence of functions $u_j : D_1 \to \mathbb{R}$ satisfying the minimal surface equation with $|Du_j| \leq 1$ on D_1 but $\sup_{D_1} |D^2 u_j| \to \infty$.*

Ex 11. *Prove that any positive harmonic function on a catenoid must be constant.*

Ex 12. *Suppose that Σ is a minimal annulus whose boundary components are circles $\{x^2 + y^2 = 1, \, z = H\}$ and $\{x^2 + y^2 = 1, \, z = -H\}$. Prove that H cannot be too large.*

Ex 13. *Suppose that Σ is a complete minimal surface so that for all R*

$$\mathrm{Area}(B_R \cap \Sigma) \leq 2\pi R^2 \, .$$

Prove that either Σ is embedded or the union of two planes.

Ex 14. *Define the divergence form operator*

$$Lu = \mathrm{div}(f(|\nabla u|^2) \, \nabla u)$$

for a nonnegative function f (e.g., $f(s) = (1+s)^{-1/2}$ gives the minimal surface equation). For which functions f does L have a strong maximum principle?

Ex 15. *Complete the proof of Theorem 1.26 by showing uniqueness of minimal graphs.*

Ex 16. *What is the Morse index of the catenoid?*

Ex 17. *Prove that the Morse index of the helicoid is infinite. Can you divide the helicoid into two stable pieces?*

11. Appendix: The Harnack inequality

We will next recall the Harnack inequality for nonnegative solutions of uniformly elliptic equations. The version that we will use is contained in theorem 8.20 of [**?**] and applies to a very general class of operators. For the next lemma, let $\mathcal{L}$ be a second-order linear differential operator on $\mathbb{R}^n$ given by

$$\mathcal{L}\, u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a^{ij}(x) \frac{\partial u}{\partial x_j} + b^i(x)\, u \right) + \sum_{i=1}^{n} c^i(x)\, \frac{\partial u}{\partial x_i} + d(x)\, u\,, \tag{190}$$

where the coefficients a^{ij}, b^i, c^i, d are measurable functions.

LEMMA 1.36. *Let $\mathcal{L}$ be a second-order linear differential operator on $\Omega \subset \mathbb{R}^n$ with bounded measurable coefficients a^{ij}, b^i, c^i, d as in* (190) *satisfying*

$$\sum_{i,j=1}^{n} a^{ij} x_i x_j \geq \lambda |x|^2 \tag{191}$$

for some $\lambda > 0$ and

$$\sum_{i,j=1}^{n} (a^{ij})^2 \leq \Lambda\,, \tag{192}$$

$$\lambda^{-2} \sum_{i=1}^{n} (|b^i(x)|^2 + |c^i(x)|^2) + \lambda^{-1} |d(x)| \leq \nu^2\,, \tag{193}$$

for some $\Lambda, \nu < \infty$. Suppose that $u \in C^0(\Omega) \cap H^1(\Omega)$ satisfies $u \geq 0$ in Ω and $\mathcal{L}u = 0$ weakly in Ω. Then, for any ball $B_{4R}(y) \subset \Omega$, we have

$$\sup_{B_R(y)} u \leq C \inf_{B_R(y)} u\,, \tag{194}$$

where $C = C(n, \frac{\Lambda}{\lambda}, \nu R) < \infty$.

By using local coordinates and a covering argument with chains of balls we can extend the Harnack inequality of Lemma 1.36 to elliptic equations on bounded domains in a Riemannian manifold.