RESEARCH STATEMENT

HAMID HEZARI

My fields of interest in mathematics are Partial Differential Equations on Riemannian manifolds and the effects of the geometry of the manifold on their solutions. In particular I am interested in applications of Microlocal Analysis in studying the Eigenvalues and Eigenfunctions of the Laplacian or more generally a Schrödinger operator. I am also interested in Mathematical Physics (Quantum Mechanics) and Geometric Analysis.

In the following I will explain my research as a graduate student at Johns Hopkins University under the supervision of Steve Zelditch, the significance of my results, and some motivations of my future research.

1. Research

1.1. **Inverse Spectral Problems: Can one hear the shape of a drum?** In the inverse spectral problem on a Riemannian manifold $(M; g)$, possibly with boundary, one determines as much as possible of the geometry of $(M; g)$ from the spectrum of its Laplacian Δ_g (with some given boundary conditions). The special inverse problem of Kac is to determine a Euclidean domain $\Omega \subset \mathbb{R}^n$ up to isometry from the spectrum of its Laplacian with Dirichlet, Neumann or more general boundary conditions. Physically, the motivation for this problem is identifying distant physical objects, such as stars or atoms, from the light or sound they emit.

1.1.1. *Inverse problems for real analytic domains.* The answer to Kac's problem is negative. Gordon, Webb and Wolpert in [GWW] found two iso-spectral plane domains which were not isometric; their examples were neither smooth nor convex. Zelditch in [Z3] proved that real analytic plane domains with symmetries of an ellipse are spectrally determined among such domains. Later, in [Z1] and [Z2], he improved this result to real analytic plane domains with only one symmetry using a different method, called the Balian-Bloch trace formula. In [HZ] we generalized these results to higher dimensions. The key tool in studying this problem was to find explicit formulas for the wave invariants associated to a bouncing ball orbit. Without stating all the conditions we showed that

Theorem **1.1.** *Real analytic domains in $\mathbb{R}^n$ with mirror symmetries across all coordinate axes are spectrally determined among such domains.*

Here are some interesting problems in this area:

Problem **1.2.** *Can we remove the symmetry assumptions? Are the convex real analytic domains spectrally determined among such domains?*

Before working on the above problems it might be beneficial to solve this potentially simpler, and yet unproven problem:

PROBLEM **1.3.** *Is there a non-trivial iso-spectral deformation of an ellipse?*

In fact if we rephrase Kac's problem as "what geometric information about the domain (or the manifold) lies in the eigenvalues of the Laplacian?" then we can ask many more interesting questions such as:

PROBLEM **1.4.** *From the knowledge of the eigenvalues can we determine whether the boundary of the domain (or the manifold itself in the boundaryless case) is real analytic?*

PROBLEM **1.5.** *From the knowledge of the eigenvalues can we determine whether the classical billiard map of the domain (or the geodesic flow of the Riemannian manifold) is ergodic?*

1.1.2. *Inverse problems for Schrödinger operators.* In my article [H2], I studied inverse spectral problems of the eigenvalue problem for the semi-classical Schrödinger operator $\hat{P} = -\frac{\hbar^2}{2}\triangle + V(x)$ on $L^2(\mathbb{R}^n)$. Here the potential $V(x)$ is a smooth function with a global minimum at $x = 0$ and $V(0) = 0$. We also add a condition to ensure that the spectrum is discrete in an energy interval $[0, \delta]$. We denote these eigenvalues in the interval $[0, \delta]$ by $\{E_j(\hbar)\}_{j=0}^m$ and call them the low-lying eigenvalues of $\hat{P}$. In [GU], Guillemin and Uribe raised the question whether we can recover the Taylor coefficients of V at $x = 0$ from the low-lying eigenvalues $E_j(\hbar)$. They established that if we assume some symmetry conditions on V, namely $V(x) = f(x_1^2, ..., x_n^2)$, then the 1-parameter family of low-lying eigenvalues, $\{E_j(\hbar) \,|\, \hbar \in (0, h_0)\}$, determines the Taylor coefficients of V at $x = 0$.

In my article [H2], I improve their results for a larger class of potentials by establishing new explicit formulas for the wave invariants at the bottom of the potential (Theorem 1.1 of [H2]). These formulas for the wave invariants are of independent interests for mathematicians. The wave invariants are defined by the coefficients of the asymptotic trace formula

$$Tr(\Theta(\hat{P})e^{\frac{-it}{\hbar}\hat{P}}) \sim \sum_{j=0}^{\infty} a_j(t)\hbar^j, \qquad \hbar \to 0,$$

where $\Theta \in C_0^\infty([0, \infty))$ is supported in $I = [0, \delta]$ and equals one in a neighborhood of 0. To my knowledge, the formulas I found in [H2] are the first explicit formulas for the wave invariants. Classically, one calculates the wave invariants by using the WKB parametrix for the propagator. But when using this method, one obtains very complicated and somehow unmanageable expressions for the wave invariants which are hopeless when trying to establish inverse spectral results. It turns out that there exists a better parametrix (rather than the WKB one) for the propagator, which significantly simplifies the calculations of the wave invariants. The construction of that kind of parametrix is the key point of my paper [H2].

As an application of the formulas in Theorem 1.1 of [H2], I improved the result of Guillemin and Uribe in [GU].

THEOREM **1.6.** *Let V be of the form*

$$V(x) = f(x_1^2, ..., x_n^2) + x_n^3 g(x_1^2, ..., x_n^2), \tag{1}$$

for some $f, g \in C^\infty(\mathbb{R}^n)$, and assume the eigenvalues of $\sqrt{Hess\, V(0)}$ are linearly independent over $\mathbb{Q}$. Then the low-lying eigenvalues of $\hat{P} = -\frac{1}{2}\hbar^2\Delta + V$ determine $\frac{\partial^{|\vec{\alpha}|}V}{\partial x^{\vec{\alpha}}}(0)$, $|\vec{\alpha}| = 2, 3$, and if $\frac{\partial^3 V}{\partial x_n^3}(0) \neq 0$, they determine all the Taylor coefficients of V at $x = 0$.

In fact when $n = 1$, Theorem 1.6 shows that with no symmetry assumption on the potential $V(x)$, the low-lying eigenvalues determine all the Taylor coefficients. This one-dimensional result was proved recently by Guillemin and Colin de Verdiere independently in [CG].

Two problems in this area still remain open:

Problem **1.7.** *Is Theorem 1.6 true without the symmetry assumption (1)?*

I think the answer to the this question is negative, as in the non-symmetric case there are too many Taylor coefficients to be determined from the wave invariants $a_j(t)$. However, this is a difficult problem to study as it needs the full knowledge of the wave invariants. In the completely resonant case i.e. $\omega_1 = ... = \omega_n$, I believe this problem is easier to study since the formulas for the wave invariants simplify.

Problem **1.8.** *Is Theorem 1.6 true without the symmetry assumption (1) in the completely resonant case $\omega_1 = ... = \omega_n$?*

I hope to find the answer to the above questions using the explicit formulas that I found in Theorem 1.1 of [H2].

1.2. **Zeros of Eigenfunctions (Nodal Lines).** Eigenfunctions of Laplacians arise in physics as modes of periodic vibration of drums and membranes. The eigenfunctions of Schrödinger operators represent stationary energy states of atoms and molecules in quantum mechanics. One interesting subject is studying the behavior of the nodal lines in the high energy (high frequency) setting, i.e. as the energy $\lambda \to \infty$. Donnelly-Fefferman proved that the total length of the nodal lines of a Laplace eigenfunction with energy λ on a real analytic surface is of order $\sqrt{\lambda}$. An interesting but difficult question is to study the distribution of the nodal lines as $\lambda \to \infty$. Do the zeros tend to a measure on our Riemannian manifold (X, g)? This problem seems far out of reach, but there are some answers if we look at the complex zeros instead of only the real zeros! Fifteen years ago, physicists were trying to connect the eigenfunctions of quantum systems and the dynamics of the classical system. They noticed that for the ergodic case the complex zeros tend to distribute uniformly in the phase space, but for the integrable systems, the zeros tend to concentrate on one-dimensional lines. This connection was recently made by Zelditch in [Z4] for the complex zeros of Laplace eigenfunctions on a manifold with ergodic geodesic flow. He showed that in the ergodic case the complex zeros tend to a unique calculable measure on $X_{\mathbb{C}}$ the complexification of X . There is hope that the results about complex zeros will eventually imply some results on the distribution of real zeros. A natural problem is to generalize the results of Zelditch for Schrödinger eigenfunctions on real analytic manifolds. This is indeed a difficult problem. As part of my research, in [H1] I studied this problem in the 1-dimensional semi-classical case and showed that as $\hbar \to 0$, the complex zeros of 1-dimensional Schrödinger eigenfunctions concentrate on lines in the complex plane called the Stokes lines. More precisely, consider the eigenvalue problem for a one-dimensional semi-classical Schrödinger operator

$$(-\hbar^2 \frac{d^2}{dx^2} + V(x))\psi(x, \hbar) = E\psi(x, h), \qquad \psi(x, \hbar) \in L^2(\mathbb{R}) \quad \hbar \to 0^+.$$

Here the potential $V(x)$ is a real polynomial of even degree with positive leading coefficient and the energy level E is fixed. These conditions quantize $\hbar$ as a decreasing sequence $\hbar_1 > \hbar_2 > ... \cdots \downarrow 0^+$. We let $\{\psi(x, h_n)\}$ be a sequence of eigenfunctions associated to $\hbar_n$, then $\psi(x,\hbar_n)$ possess analytic continuations $\psi(z,\hbar_n)$ to $\mathbb{C}$. Our interest is in the distribution of complex zeros of $\psi(z,\hbar_n)$ as $h_n \to 0^+$. We define the discrete measure $Z_{\hbar_n}$ by

$$Z_{\hbar_n} = \hbar_n \sum_{\{z\in\mathbb{C}:\, \Psi(z,\hbar_n)=0\}} \delta_z.$$

We would like to study the limits of weak convergent subsequences of the sequence $\{Z_{\hbar_n}\}$ as $n \to \infty$. We will call these weak limits *the zero limit measures.*

THEOREM **1.9.** *Let $V(x)$ be a real polynomial of even degree with positive leading coefficient. Then every weak limit Z of the sequence $\{Z_{\hbar_n}\}$ is of the form*

$$Z = \frac{1}{\pi}\sqrt{|V(z)-E|}\,|d\gamma|, \qquad \textit{where} \quad |d\gamma| = |\gamma'(t)|\,dt, \tag{2}$$

where γ is a union of finitely many smooth connected curves γ_m in the plane. For each γ_m there exists a constant c_m, a canonical domain D_m and a turning point z_m on the boundary of D_m such that γ_m is given by

$$\gamma_m = \big\{z \in D_m; \quad \Re(S(z_m,z)) = c_m\big\}$$

where $S(z_m,z) = \int_{z_m}^{z} \sqrt{V(\zeta)-E}\,d\zeta$ and the integral is taken along any path in D_m joining z_m to z.

This theorem shows that if Z in (2) is the limit of a subsequence $\{Z_{\lambda_{n_k}}\}$, then the complex zeros of $\{\psi(z,\hbar_{n_k})\}$ tend to concentrate on γ as $k \to \infty$ and in the limit they cover γ. The factor $|\sqrt{V(z)-E}|$ indicates that the limit distribution of the zeros on γ is measured by the Agmon metric. We call the curves γ_m the *zeros lines* of the limit Z. The next question after seeing Theorem 1.9 is "what are all the possible zero limit measures and corresponding zero lines for a given polynomial $V(x)$ and a given energy level E?" In [H1] I found all the possible zero lines for some one-well and double-well potentials. Figures (1) and (2) illustrate the zeros lines that I found for some potentials.

1.2.1. *Grauert Tubes for Schrödinger Operators.* I expect to extend my work to the distribution of complex zeros of Schrödinger eigenfunctions on a real analytic manifold of any dimension. This requires developing some complex geometry and associating an analogue of the Grauert tubes (which was used by Zelditch for Laplacian) to a Schrödinger operator. In [BW], Bruhat-Whitney showed that for a real analytic Riemannian manifold (X,g) of dimension n, there always exists a thickening $X_{\mathbb{C}}$ of X called the complexification of X, which is a complex manifold that contains X as a totally real sub-manifold. In [GS], Guillemin-Stenzel showed that there exists a unique strictly pluri-subharmonic function ρ on $X_{\mathbb{C}}$ such that $\rho^{-1}(0) = X$, $\sqrt{\rho}$ satisfies the complex Monge Ampere equation on $X_{\mathbb{C}}\backslash X$, i.e. $(\frac{\sqrt{-1}}{2}\partial\bar{\partial}\sqrt{\rho})^n = 0$ and if we equip $X_{\mathbb{C}}$ with the Kähler metric $\omega = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\rho$ then the inclusion map $i : (X,g) \to (X_{\mathbb{C}},\omega)$ is an isomorphic embedding. The Grauert tubes are defined by $X_\epsilon = \{z \in X_{\mathbb{C}} |\ \rho(z) < \epsilon\}$. Boutet de Monvel in [Bou] used the analytic continuation in the time and space variables of the kernel of the wave group to obtain $\varphi^{\mathbb{C}}_{\lambda_j}$, the analytic continuation of the Laplace eigenfunctions φ_{λ_j} of (X,g) to X_ϵ. Then Zelditch in [Z4] showed that if the geodesic flow of X is ergodic, the zeros current Z_{λ_j} of $\varphi^{\mathbb{C}}_{\lambda_j}$, as $\lambda_j \to \infty$, is asymptotic to $(\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\sqrt{\rho})\,\lambda_j$. When one wants to study the complex zeros of the eigenfunctions of a semi-classical Schrödinger operator then one has to study the appropriate Grauert tubes in

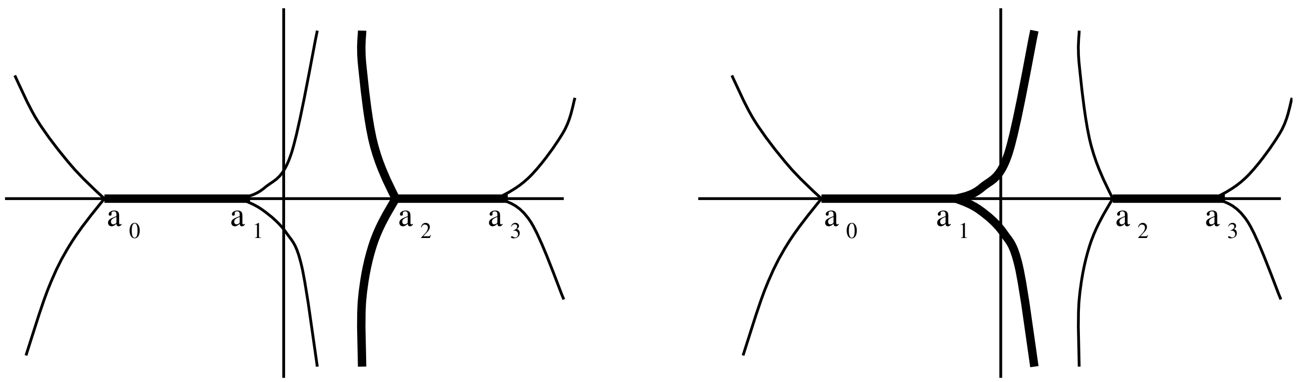
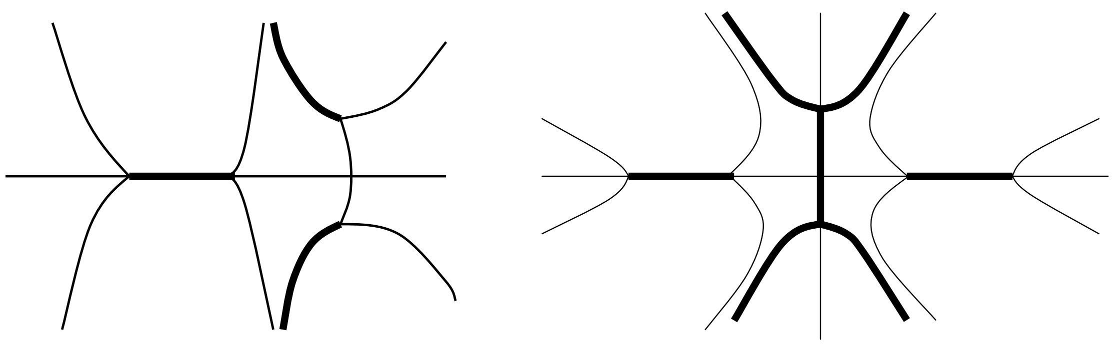


FIGURE 1. The zero lines of a general non-symmetric quartic oscillator given by $V(x) - E = (x - a_0)(x - a_1)(x - a_2)(x - a_3)$. The thick lines are the zeros lines, and thin lines are the Stokes lines. The left figure shows the limiting zero lines when the eigenvalues are quantized around the well (a_0, a_1), and the one on the right corresponds to quantization around the well (a_2, a_3).



$V(x) - E = (x^2 - a^2)(x^2 + bx + c)$ $\qquad$ $V(x) - E = (x^2 - a^2)(x^2 - b^2)(x^2 + c^2)$

FIGURE 2.

this new setting. More precisely, let us fix an energy level E and let C be a classically allowed region on X, i.e. on C, $V(x) < E$. Then we obtain a unique pluri-subharmonic function ρ_V on $C_{\mathbb{C}}$ satisfying all the properties as ρ above except this time $i : (C, g_V) \to (C_{\mathbb{C}}, \omega_V)$ is an isomorphic embedding, where $g_V = (E - V(x))g$ is the associated Agmon metric on C and $\omega_V = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\rho_V$. So here the Grauert tubes are defined by $C_\epsilon = \{z \in C_{\mathbb{C}} | \; \rho_V(z) < \epsilon\}$. Finally we should analytically continue the Schrödinger eigenfunctions to these tubes and study the limit distribution of the complex zeros there.

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Mathematics Department, Johns Hopkins University, Baltimore, MD, 21218
E-mail address: hhezari@math.jhu.edu