RESEARCH STATEMENT
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1. Introduction

Minimal surfaces are defined as immersed surfaces that are critical points for the area functional. Using the so called first variation formula, one can easily show that every minimal surface has mean curvature identically zero. A subclass of such surfaces, area minimizing surfaces, satisfy the natural property of being the surface with least area given a prescribed boundary. There are an abundance of non-trivial examples that exist in $\mathbb{R}^3$, even when one imposes the additional condition of embeddedness (i.e. without self-intersection). Classic examples of embedded minimal surfaces include the plane, catenoid, and helicoid.

2. Classifying Minimal Surfaces

The classification of embedded minimal surfaces is an old problem with a rich history. Recently, there has been dramatic progress on a number of fronts, but some fundamental questions remain open. I will describe my recent results, explain the consequences and significance, and briefly discuss the proof.

2.1. **Conformality Results.** We say a surface Σ has *finite topology* if it is homeomorphic to a compact Riemann surface with a finite number of points removed. When Σ is complete, each removed point corresponds to an end of Σ. Let $\mathcal{M}_1$ denote the space of complete, embedded minimal surfaces in $\mathbb{R}^3$ with finite topology and one end. Relying heavily on the lamination theory of Colding and Minicozzi [8], Meeks and Rosenberg in [18] completely classify simply connected surfaces in $\mathcal{M}_1$.

Theorem 2.1. *(Theorem 0.1 in* [18]*, Theorem 1.4 in* [1]*) The only complete, properly embedded minimal disks in $\mathbb{R}^3$ are the plane and the helicoid.*

Jacob Bernstein and I, in [1], provide an alternate proof to Theorem 2.1, using directly the new estimates and structural results developed by Colding and Minicozzi in their series of papers [5, 6, 7, 8]. Moreover, coupling these tools with a Colding-Minicozzi compactness result for higher genus surfaces from [9], in [3] we prove an important conformality result for all surfaces in $\mathcal{M}_1$.

Theorem 2.2. *(Theorem 1.1 in* [3]*) $\Sigma \in \mathcal{M}_1$ is conformally a punctured, compact Riemann surface. Moreover, after a rotation, the height differential, dh, extends meromorphically over the puncture with a double pole, as does the meromorphic one form $\frac{dg}{g}$.*

Theorem 2.2 completes the understanding of the conformal structure of minimal surfaces with finite topology. In [19], Meeks and Rosenberg prove conformality results for properly embedded minimal surface of finite topology and two or more ends. Using their work, Corollary 0.13 of [10], and Theorem 2.2, we have the following:

Corollary 2.3. *Every complete, embedded minimal surface of finite topology in $\mathbb{R}^3$ is conformal to a compact Riemann surface with a finite number of punctures.*

Recall that every immersed minimal surface in $\mathbb{R}^3$ admits a parametrization using the so called Weierstrass data (M, g, dh), where M is a Riemann surface, g is the stereographic projection of the Gauss map, and dh is the height differential. The helicoid has Weierstrass data $g = e^{i\alpha z}$, $dh = dz$ for $z \in \mathbb{C}, \alpha \in \mathbb{R}$; thus $\frac{dg}{g} - i\alpha dh$ is identically zero. For $\Sigma \in \mathcal{M}_1$, Theorem 2.2, the Weierstrass representation, and embeddedness imply that at the puncture the Weierstrass data is asymptotic to that of a helicoid. Precisely,

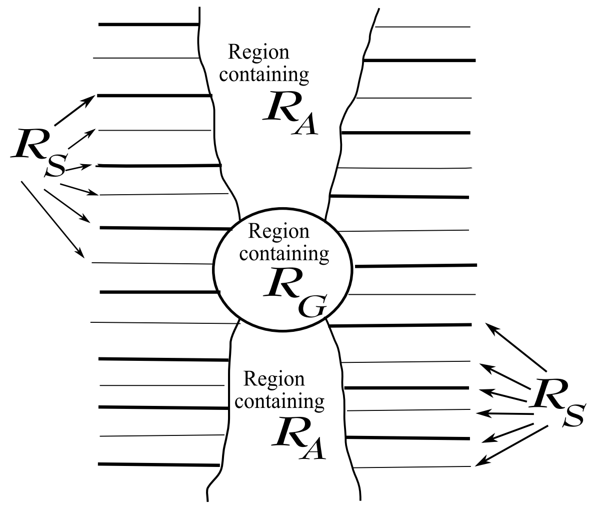

FIGURE 1. A rough sketch of the three regions in the decomposition of Σ as outlined.

Corollary 2.4. *(Corollary 1.3 in [3]) There exists an $\alpha \in \mathbb{R}$ so $\frac{dg}{g} - i\alpha dh$ has holomorphic extension over the puncture, with a zero at the puncture. Equivalently, after possibly translating parallel to the x_3-axis, in an appropriately chosen neighborhood of the puncture, Γ, $g(p) = \exp(i\alpha z(p) + F(p))$ where $F : \Gamma \to \mathbb{C}$ extends holomorphically over the puncture with a zero there and $z = x_3 + ix_3^*$ is a holomorphic coordinate on Γ. (Here x_3^* is the harmonic conjugate of x_3 and is well defined in Γ.)*

As a consequence of this we may appeal to [14] where the behavior of annular ends with this type of Weierstrass data are studied. In particular, Hauswirth, Perez and Romon show that such an end is C^0-asymptotic[1] to a (vertical) helicoid H. They examine annular ends with Weierstrass data as described in Corollary 2.4 and prove the asymptotic result under the assumptions that the end is conformal to a punctured disk and the data satisfy a certain flux condition. Theorem 2.2 provides the necessary conformality, and appealing to Stokes' Theorem shows the flux condition is automatically satisfied for all $\Sigma \in \mathcal{M}_1$. Thus,

Corollary 2.5. *If $\Sigma \in \mathcal{M}_1$ is non-flat then Σ is C^0-asymptotic to some helicoid.*

2.2. **Outline of Proof.** The proof of Theorem 2.2 requires we first prove a strong structural decomposition. We show Σ can be decomposed into three regions: $\mathcal{R}_G$ contains the genus and is compact, $\mathcal{R}_A$ contains the "axis" which includes points of large curvature, and $\mathcal{R}_S$ contains two strictly spiraling multi-valued graphs. (See Figure 1.) As examples consider the helicoid and the genus one helicoid of [21]. (See Figure 2.) The helicoid contains a vertical axis with the points of large curvature, $\mathcal{R}_A$. Removing this axis leaves two spiraling disks which are locally graphs; this is the region $\mathcal{R}_S$. Obviously, for the helicoid $\mathcal{R}_G$ is empty. In the case of the genus one helicoid, let $\mathcal{R}_G$ surround the genus and then choose $\mathcal{R}_A$ as the components of $\Sigma \backslash \mathcal{R}_G$ in a sufficiently large infinite cylinder centered on the z-axis.

In $\Sigma \backslash \mathcal{R}_G$, the strictly spiraling region coupled with an application of Rado's theorem guarantee that $|\nabla_\Sigma x_3|$ never vanishes and thus $z = x_3 + ix_3^*$, where x_3^* is the harmonic conjugate to x_3, is an injective holomorphic map $z : \Sigma \backslash \mathcal{R}_G \to \mathbb{C}$. By using a Picard type argument on the level sets of the composition of z with the stereographic projection of the Gauss map, g, we establish Theorem 2.2.

[1] i.e. for any $\epsilon > 0$ there exists $R_\epsilon > 0$ so that the part of the end outside of $B_{R_\epsilon}(0)$ has Hausdorff distance to H less than ϵ

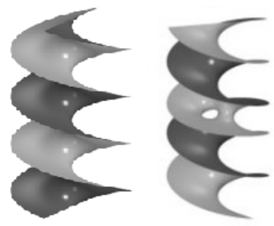

FIGURE 2. (a) A helicoid, (b) A genus one helicoid

3. Distortions of the Helicoid

In [5, 6, 7, 8], Colding and Minicozzi give a complete description of the structure of embedded minimal disks in $\mathbb{R}^3$. Roughly speaking, they show that any such surface is, depending on the curvature, either modeled on a plane (i.e. is nearly graphical) or is modeled on a helicoid (i.e. is two multi-valued graphs glued together along an axis). In the latter case, the distortion from the helicoid can be quite large. For instance, in [20], Meeks and Weber "bend" the helicoid; that is, they construct minimal disks where the axis is an arbitrary $C^{1,1}$ curve.

A more serious example of distortion is given by Colding and Minicozzi in [11]. There they construct a sequence of minimal disks modeled on the helicoid, but where the the ratio between the scales (i.e. the measure of the tightness of the spiraling of the multi-graphs) at different points of the axis is arbitrarily large. Importantly, such large distortions are, in some sense, global. Indeed, near points of large curvature a minimal disk looks like a piece of the helicoid with small distortion.

For Σ minimal, we define a pair $(y, s) \in \Sigma \times \mathbb{R}^+$ to be a *blow-up pair* if

$$\sup_{\Sigma \cap B_s(y)} |A|^2 \leq 4|A|^2(y) = 4C^2 s^{-2}. \tag{3.1}$$

In [6], Colding and Minicozzi established that near $y \in \Sigma$ a multi-valued graph is formed. We use a compactness argument to demonstrate that an embedded minimal disk with boundary in B_R and blow-up pair (y, s) is bi-Lipschitz, on scale s, to a piece of a helicoid. Precisely,

Theorem 3.1. *(Theorem 1.5 in* [1]*) Given $\epsilon, R > 0$ there exist $s > 0$ and $R' \geq R$ so: Suppose $0 \in \Sigma'$ is a properly embedded minimal disk with $\Sigma' \subset B_{R'}(0)$, $\partial\Sigma' \subset \partial B_{R'}(0)$, and $(0, s)$ a blow-up pair. Then there exists Ω, a subset of some helicoid, so that Σ, the component of $\Sigma' \cap B_R$ containing 0, is bi-Lipschitz with Ω with the Lipschitz constant in $((1+\epsilon)^{-1}, 1+\epsilon)$.*

Using the example provided by Colding and Minicozzi in [11] and a compactness argument, we show that a result like Theorem 3.1 cannot hold on the outer scale R or even on smaller scales.

Theorem 3.2. *(Theorem 0.1 in* [2]*) Given $\epsilon > 0$, $1 > \Omega > 0$ and $1/2 > \gamma \geq 0$ there exists an embedded minimal disk $0 \in \Sigma$ with $\partial\Sigma \subset \partial B_R$ and $(0, s)$ a blow-up pair so: The component of $B_{\Omega R^{1-\gamma} s^{\gamma}} \cap \Sigma$ containing 0 is not bi-Lipschitz to a piece of the helicoid with Lipschitz constant in $((1+\epsilon)^{-1}, 1+\epsilon)$.*

As an application of their work on the structure of disks, Colding and Minicozzi proved a compactness result for sequences of embedded minimal disks $0 \in \Sigma_i \subset \mathbb{R}^3$ as long as $\partial\Sigma_i \subset \partial B_{R_i}$ and $R_i \to \infty$. In particular, they show there are only two options. Either such a sequence contains a sub-sequence converging smoothly on compact sets to a complete embedded minimal disk or, if the curvature is unbounded in some compact subset of $\mathbb{R}^3$, the convergence is (in a certain sense, see [8] for details) to a singular minimal lamination of parallel planes. The surfaces constructed by Colding and Minicozzi in [11] show that the condition that the boundaries of the surface go to infinity is essential, i.e this compactness result is global in nature. In a similar vein, the result depends very strongly on the ambient geometry of the three-manifold. In particular, in the proof of their compactness result, Colding and Minicozzi rely heavily on a flux argument (the details of which are in [12]). They use that the coordinate functions of $\mathbb{R}^3$ restrict to harmonic functions on minimal $\Sigma \subset \mathbb{R}^3$, a fact that generalizes only to certain other highly symmetric three-manifolds.

One of the most important problems in this area is determining when a Colding and Minicozzi type of compactness result (or indeed any compactness result) extends to surfaces embedded in more arbitrary three-manifolds. Understanding precisely the best scale for which the Lipschitz approximation holds (for which Theorem 3.2 gives an upper bound) may be an important tool to establish removable singularities theorems for minimal laminations in arbitrary Riemannian manifolds. In turn, such results could prove key to proving more general compactness theorems.

4. Current and Future Work

4.1. **Continuing to Classify Surfaces in $\mathcal{M}_1$.** By constructing an embedded, genus one helicoid in [21] (see Figure 2), Hoffman, Weber, and Wolf confirmed that there exists a surface in $\mathcal{M}_1$ with non-trivial genus. Their surface contains a vertical and a horizontal axis, is homeomorphic to a torus with a puncture, and is asymptotic to a helicoid, H. In that same paper, they conjecture that any surface with these properties is actually unique, up to scaling.

Conjecture 4.1. *(Conjecture 1 in* [21]*) There exists a unique properly embedded minimal surface Σ that contains exactly two straight lines (the x- and z-axes), has genus equal to one, and has one end asymptotic to the helicoid.*

The moduli space of surfaces in $\mathcal{M}_1$ with fixed genus g, $\mathcal{M}_1^g$, is not compact; simply consider rescalings of the helicoid. Under a sequence of rescalings, the helicoids converge to a foliation of $\mathbb{R}^3$ by parallel planes away from a singular axis. In [9], Colding and Minicozzi address, among other things, exactly how $\mathcal{M}_1^g$ fails to be compact. That is, they describe the geometry of surfaces that lie on the boundary of this moduli space. Their work, as well as Conjecture 4.1, gives rise to many questions.

Problem 4.2. *What subspaces of $\mathcal{M}_1^g$ are compact?*

To consider compactness on any of these new spaces will require understanding how the Weierstrass data and the conformal structure influence the global and local geometry of the surfaces. Answers to questions like the following will be critical to determining new compactness results.

Problem 4.3. *How do the coefficients of the function f (as defined in Corollary 2.4) influence the asymptotic geometry of Σ?*

Problem 4.4. *How does the conformal structure of Σ influence the geometry of the region containing the genus?*

Some relationships are already well known. For instance, if Σ is a genus-one helicoid containing both the x- and z-axes, then simple complex analysis arguments on the symmetries of Σ show that Σ must be conformal to a rhombic torus and the coefficients of the function f described in Corollary 2.4 are purely imaginary.

I wish to consider the answers to Problems 4.3 and 4.4 in the more general case, without the symmetry assumptions. In addition, I expect that the techniques developed to answer Problems 4.2, 4.3, and 4.4 will ultimately lead to a proof of or counterexample to Conjecture 4.1.

4.2. **Calabi Type Questions.** Central to the study of surfaces in $\mathbb{R}^3$ is the existence of complete, non-compact surfaces Σ constrained to lie in a specific region. When Σ is minimal, these problems are commonly known as the Calabi-Yau conjectures [4]. As previously mentioned, Colding and Minicozzi prove a version of these conjectures is true in [10].

Theorem 4.5. *(Corollary 0.13 in* [10]*) A complete minimal surface of finite topology embedded in* $\mathbb{R}^3$ *must be properly embedded.*

I am considering a question of this type.

Problem 4.6. *Is an embedded minimal surface of finite genus properly embedded?*

Meeks, Perez, and Ros [17] answer this question with the additional assumption of locally bounded curvature. I expect such a Calabi-Yau result is true, even when removing this assumption.

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