HELICOID-LIKE MINIMAL DISKS AND UNIQUENESS

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Abstract. We show that an embedded minimal disk in $\mathbb{R}^3$ with large curvature is bi-Lipschitz with a piece of a helicoid. Additionally, a simplified proof of the uniqueness of the helicoid is provided.

1. Introduction

This paper gives a condition for an embedded minimal disk to be bi-Lipschitz to a piece of a helicoid. Namely, if such a disk is inside a ball (with boundary on the ball) and has large curvature at the center relative to nearby points, then, in a smaller ball, it is bi-Lipschitz to a piece of a helicoid. Moreover, the Lipschitz constant can be chosen as close to 1 as desired. This is a sharpening of a result of Colding and Minicozzi showing that, in a rough sense, such a disk looks like a piece of the helicoid. Our result follows from the fact that the only complete, non-flat, properly embedded minimal disk (PEMD) in $\mathbb{R}^3$ is the helicoid, for which we supply a proof. The initial proof, given by Meeks and Rosenberg in [17], depends crucially on the lamination theory and one-sided curvature estimate of Colding and Minicozzi (see [5]). Instead of appealing to the lamination theory, we make direct use of the results of Colding and Minicozzi on the existence of multivalued graphs in embedded minimal disks, as found in [6, 7, 8, 5]; note that [9] contains a good non-technical overview. Applying a result of [4] to these multivalued graphs, we approximate them by pieces of helicoids, giving explicit asymptotic behavior and geometric rigidity. In [1], these techniques will be further studied.

In their paper, Meeks and Rosenberg first use the lamination theory to show that (after a rotation) a homothetic blow-down of a non-flat complete PEMD, Σ, is, away from some Lipschitz curve, a foliation of flat parallel planes transverse to the x_3-axis. This gives, in a weak sense, that the surface is asymptotic to a helicoid, which they use to conclude that the Gauss map of Σ omits the north and south poles. The asymptotic structure combined with a result on parabolicity of Collin, Kusner, Meeks and Rosenberg [11], is then used to show that Σ is conformally equivalent to $\mathbb{C}$. Finally, they look at level sets of the log of the Gauss map and use a Picard type argument to show that this holomorphic map does not have an essential singularity at ∞ and in fact is linear. Using the Weierstrass representation, they conclude that Σ is the helicoid.

The explicit asymptotics in our paper allow for a more direct approach. We show Σ contains a central "axis" of large curvature away from which it consists of two multivalued graphs spiraling together, one strictly upward, the other downward. This is the structure of the helicoid and more generally, at least away from a compact set, the structure of the (known) embedded genus one helicoid(s) i.e. the construction of Weber, Hoffman and Wolf, [13], and that of Hoffman and White,

[14], and, indeed, of any symmetric genus one helicoid (see [15]). Moreover, this is the behavior of any complete, non-flat PEMD:

Theorem 1.1. *There exist subsets of* Σ, $\mathcal{R}_A$ *and* $\mathcal{R}_S$, *with* $\Sigma = \mathcal{R}_A \cup \mathcal{R}_S$ *where* $\mathcal{R}_S$ *can be written as the union of two (oppositely oriented) multivalued graphs* u^1 *and* u^2 *with non-vanishing angular derivative. Further, there exists* $\epsilon_0 > 0$ *such that on* $\mathcal{R}_A$, $|\nabla_\Sigma x_3| \geq \epsilon_0$.

Remark 1.2. Here u^i multivalued means that it can be decomposed into N-valued ϵ-sheets (see Definition 2.1) with varying center. The angular derivative is then with respect to the obvious polar form on each of these sheets. For simplicity we will assume throughout that both u^i are ∞-valued.

In order to establish this decomposition we first use the explicit asymptotics to get the strict spiraling in $\mathcal{R}_S$. An application of the proof of Rado's theorem (see [18]) then gives non-vanishing of $|\nabla_\Sigma x_3|$ on $\mathcal{R}_A$ and, by a Harnack inequality, the uniform lower bound. Crucially,

Proposition 1.3. *On* Σ, $\nabla_\Sigma x_3 \neq 0$ *and, for all* $c \in \mathbb{R}$, $\Sigma \cap \{x_3 = c\}$ *consists of exactly one properly embedded smooth curve.*

This implies that $z = x_3 + i x_3^*$ is a holomorphic coordinate on Σ. By looking at the stereographic projection of the Gauss map, g, in $\mathcal{R}_S$ we show that z maps onto $\mathbb{C}$ and so Σ is conformally the plane. This follows from the control on the behavior of g due to strict spiraling. Indeed, away from a small neighborhood of $\mathcal{R}_A$, Σ is conformally the union of two closed half-spaces with $\log g = h$ providing the identification. It then follows that h is also a conformal diffeomorphism which gives the result proved by Meeks and Rosenberg (Theorem 0.1 in [17]).

Theorem 1.4. *The only complete PEMDs in* $\mathbb{R}^3$ *are the plane and the helicoid.*

Finally, the local result will follow from compactness (compare with Proposition 2 of [16]):

Theorem 1.5. *Given* $\epsilon, R > 0$ *there exists* $R' \geq R$ *so: Suppose* $0 \in \Sigma'$ *is a PEMD with* $\Sigma' \subset B_{R's}(0)$, $\partial\Sigma' \subset \partial B_{R's}(0)$, *and* $(0, s)$ *a blow-up pair (see section 2.2). Then there exists* Ω, *a subset of a helicoid, so that* Σ, *the component of* $\Sigma' \cap B_{Rs}$ *containing 0, is bi-Lipschitz with* Ω, *and the Lipschitz constant is in* $(1-\epsilon, 1+\epsilon)$.

2. Preliminaries

To study Σ we rely heavily on the structural results of Colding and Minicozzi regarding embedded minimal disks. Much of this can be found in the series of papers [6, 7, 8, 5], with more technical analysis in [3]. For a more general overview of the results, the interested reader should consult the survey [4].

2.1. **Notation.** Throughout, Σ will be a complete, non-flat, PEMD. Let

$$\mathbf{C}_\delta(y) = \left\{x : (x_3 - y_3)^2 \leq \delta^2\left((x_1 - y_1)^2 + (x_2 - y_2)^2\right)\right\} \subset \mathbb{R}^3 \tag{2.1}$$

be a cone and set $\mathbf{C}_\delta = \mathbf{C}_\delta(0)$. We denote a polar rectangle as follows:

$$S^{\theta_1,\theta_2}_{r_1,r_2} = \{(\rho, \theta) \mid r_1 \leq \rho \leq r_2, \theta_1 \leq \theta \leq \theta_2\}. \tag{2.2}$$

For a real-valued function, u, defined on a polar domain $\Omega \subset \mathbb{R}^+ \times \mathbb{R}$, define the map $\Phi_u : \Omega \to \mathbb{R}^3$ by $\Phi_u(\rho, \theta) = (\rho\cos\theta, \rho\sin\theta, u(\rho, \theta))$. In particular, if u is defined on

$S^{\theta_1,\theta_2}_{r_1,r_2}$, then $\Phi_u(S^{\theta_1,\theta_2}_{r_1,r_2})$ is a multivalued graph over the annulus $D_{r_2}\backslash D_{r_1}$. We define the separation of the graph u by $w(\rho,\theta) = u(\rho,\theta+2\pi)-u(\rho,\theta)$. Thus, $\Gamma_u := \Phi_u(\Omega)$ is the graph of u, and Γ_u is embedded if and only if $w \neq 0$.

Recall that u satisfies the minimal surface equation if:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0. \tag{2.3}$$

We will require that such u also satisfies the following flatness condition:

$$|\nabla u| + \rho|\text{Hess}_u| + 4\rho\frac{|\nabla w|}{|w|} + \rho^2\frac{|\text{Hess}_w|}{|w|} \leq \epsilon < \frac{1}{2\pi}. \tag{2.4}$$

Note that if w is the separation of a u satisfying (2.3) and (2.4), then w satisfies a uniformly elliptic equation. Thus, if Γ_u is embedded then w has pointwise gradient bounds and a Harnack inequality.

2.2. **Initial Sheets.** In Colding and Minicozzi's work, multivalued minimal graphs form the basic building block used to study the structure of minimal surfaces. We also make heavy use of the properties of such graphs, which we normalize as follows:

Definition 2.1. A multivalued minimal graph Σ_0 is an *N-valued (ϵ-)sheet (centered at 0 on the scale 1)*, if $\Sigma_0 = \Gamma_u$ and u, defined on $S^{-\pi N,\pi N}_{1,\infty}$, satisfies (2.3), (2.4), $\lim_{\rho\to\infty}\nabla u(\rho,0) = 0$, and $\Sigma_0 \subset \mathbf{C}_\epsilon$.

Using Simons' inequality, Corollary 2.3 of [3] shows that on the one-valued middle sheet of a 2-valued graph satisfying (2.4), the hessian of u has faster than linear decay. For an ϵ-sheet, Γ_u, this implies a Bers like result on asymptotic tangent planes. Indeed, the normalization at ∞ gives gradient decay,

$$|\nabla u| \leq C\epsilon\rho^{-5/12}. \tag{2.5}$$

We now give a condition for the existence of ϵ-sheets. Roughly, all that is required is a point with large curvature relative to nearby points. Precisely,

Definition 2.2. The pair (y, s), $y \in \Sigma$, $s > 0$, is a *(C) blow-up pair* if

$$\sup_{\Sigma\cap B_s(y)} |A|^2 \leq 4|A|^2(y) = 4C^2 s^{-2}. \tag{2.6}$$

Having a blow-up pair forces the surface to spiral nearby (see Theorem 0.4 of [7]). In particular, after a suitable rotation we obtain an ϵ-sheet. For a more thorough treatment, see Theorem 7.1 in the Appendix.

Once we have one ϵ-sheet, we can use the one-sided curvature estimates of Theorem 0.2 of [5] to extend the graph (and (2.4)) from an ϵ-sheet to a narrow cone. Specifically, there is a curvature bound on embedded minimal disks close to, but on one side of, a flat minimal surface. Thus, using the initial ϵ-sheet as such a flat surface implies that in a cone all pieces of Σ are graphs. A barrier argument then shows that there are only two such pieces. Namely, by Theorem I.0.10 of [5], the parts of Σ that lie in between an ϵ-sheet make up a second multivalued graph. Furthermore, one-sided curvature gives gradient estimates which, coupled with the estimates we get given enough sheets, reveal that this graph actually contains an ϵ-sheet. Thus, around a blow-up point, Σ consists of two ϵ-sheets spiraling together.

We now make the last statement precise. Suppose u is defined on $S^{-\pi N-3\pi,\pi N+3\pi}_{1/2,\infty}$ and Γ_u is embedded. We define E to be the region over $D_\infty \backslash D_1$ between the top and bottom sheets of the concentric subgraph of u. That is:

$$E = \{(\rho\cos\theta, \rho\sin\theta, t) : \\ 1 \leq \rho \leq \infty, -2\pi \leq \theta < 0, u(\rho, \theta - \pi N) < t < u(\rho, \theta + (N+2)\pi\}. \tag{2.7}$$

Using Theorem I.0.10 of [5], Theorem 7.1, and one-sided curvature, we have:

Theorem 2.3. *Given $\epsilon > 0$ sufficiently small, there exist $C_1, C_2 > 0$ so: Suppose $(0,s)$ is a C_1 blow-up pair. Then there exist two 4-valued ϵ-sheets $\Sigma_i = \Gamma_{u_i}$ $(i = 1,2)$ on the scale s which spiral together (i.e. $u_1(s,0) < u_2(s,0) < u_1(s,2\pi)$). Moreover, the separation over ∂D_s of Σ_i is bounded below by $C_2 s$.*

Remark 2.4. We refer to Σ_1, Σ_2 as (ϵ-)blow-up sheets *associated with* (y,s).

Proof. Choose $\epsilon_0 > 0$ and N_0 as in Theorem I.0.10 (of [5]). For $\epsilon < \epsilon_0$ choose N_ϵ, δ_ϵ as in the proof of Theorem 7.1. With $N - 6 = \max\{N_\epsilon + 4, N_0\}$ denote by C_1', C_2' the constants given by Theorem 7.1. Thus, if $(0,r)$ is a C_1' blow-up pair then there exists an N-valued ϵ-sheet $\Sigma_1' = \Gamma_{u_1'}$ on scale r inside of Σ. Applying Theorem I.0.10 to u_1', we see that $\Sigma \cap E \backslash \Sigma_1'$ is given by the graph of a function u_2' defined on $S^{-\pi N_\epsilon - 4\pi, \pi N_\epsilon + 4\pi}_{2r,\infty}$. In particular, for u_2' on $S^{-4\pi,4\pi}_{2e^{N_\epsilon}r,\infty}$ we have (2.4) as long as we can control $|\nabla u_2'|$. But here we use one-sided curvature (and the ϵ-sheet Σ_1'). Namely, given $\alpha = \min\{\epsilon/2, \delta_\epsilon\}$, one-sided curvature estimates allow us to choose $\delta_0 > 0$ so that in the cone $\mathbf{C}_{\delta_0}$ (and outside a ball) Σ is graphical with gradient less than α. By (2.5), there exists $r_1 > 0$ such that $|\nabla u_1'| \leq \delta_0$ on $S^{-5\pi,5\pi}_{r_1,\infty}$ and this 5-valued graph is contained in $\mathbf{C}_{\delta_0} \backslash B_{r_1}$. Moreover, since five sheets of u_1' are inside of $\mathbf{C}_{\delta_0}$, the four concentric sheets of u_2' are also in that cone. Set $\gamma = \max\{2e^{N_\epsilon}, 1\}$. Let u_1 and u_2 be given by restricting u_1' and u_2' to $S^{-4\pi,4\pi}_{\gamma r_1,\infty}$ and define $\Sigma_i = \Gamma_{u_i}$.

Set $C_1 = \gamma C_1'$, so if $(0,s)$ is a C_1 blow-up pair then Σ_i will exist on scale s. Integrating (2.4), the lower bound C_2' gives a lower bound on initial separation of Σ_1. We find C_2 by noting that if the initial separation of Σ_2 was too small there would be two sheets between one sheet of Σ_1. □

2.3. **Blow-up Pairs.** Since Σ is not a plane, we can always find at least one blow-up pair (y,s). We then use this initial pair to find a sequence of blow-up pairs forming an "axis" of large curvature. The key results we need are Lemma 5.1 of [7], which says that as long as curvature is large enough in some ball we can find a blow-up pair in the ball, and Corollary III.3.5 of [8], which guarantees points of large curvature above and below blow-up points. Colding and Minicozzi, in [10], provide a good overview of this process of decomposing Σ into blow-up sheets. The main result is the following (see Lemma 2.5 of [10]):

Theorem 2.5. *For $1/2 > \gamma > 0$ and $\epsilon > 0$ both sufficiently small, let C_1 be given by Theorem 2.3. Then there exists $C_{in} > 4$ and $\delta > 0$ so: If $(0,s)$ is a C_1 blow-up pair then there exist (y_+, s_+) and (y_-, s_-), C_1 blow-up pairs, with $y_\pm \in \Sigma \cap B_{C_{in}s} \backslash (B_{2s} \cup \mathbf{C}_\delta)$, $x_3(y_+) > 0 > x_3(y_-)$, and $s_\pm \leq \gamma |y_\pm|$.*

Hence, given a blow-up pair, we can iteratively find a sequence of blow-up pairs ordered by height and lying outside of a cone, with distance between subsequent pairs bounded by a fixed multiple of the scale.

3. Asymptotic Helicoids

Lemma 14.1 of [4] and the gradient decay (2.5) shows that ϵ-sheets can be approximated by a combination of planar, helicoidal, and catenoidal pieces. Precisely, there is a "laurent expansion" for the almost holomorphic function $u_x - iu_y$. This result allows us to bound the oscillation on broken circles $C(\rho) := S^{-\pi,\pi}_{\rho,\rho}$ of u_θ, which yields asymptotic lower bounds for u_θ.

Lemma 3.1. *Given Γ_u, a 3-valued ϵ-sheet on scale 1, set $f = u_x - iu_y$. Then for $r_1 \geq 1$ and $\zeta = \rho e^{i\theta}$ with $(\rho, \theta) \in S^{-\pi,\pi}_{2r_1,\infty}$*

$$f(\zeta) = c\zeta^{-1} + g(\zeta) \tag{3.1}$$

where $c = c(r_1, u) \in \mathbb{C}$ and $|g(\zeta)| \leq C_0 r_1^{-1/4}|\zeta|^{-1} + C_0\epsilon r_1^{-1}|w(r_1, -\pi)|$.

Using this approximation result we now bound the oscillation.

Lemma 3.2. *Suppose Γ_u is a 3-valued ϵ-sheet on scale 1. Then for $\rho \geq 2$, there exists a universal C so:*

$$\operatorname*{osc}_{C(\rho)} u_\theta \leq C\rho^{-1/4} + C\epsilon|w(\rho, -\pi)|. \tag{3.2}$$

Proof. Using Lemma 3.1 and the identification $u_\theta(\rho, \theta) = -\text{Im}\, \zeta f(\zeta)$ for $\zeta = \rho e^{i\theta}$, we compute:

$$\begin{aligned} \operatorname*{osc}_{C(\rho)} u_\theta &= \sup_{|\zeta|=\rho} \text{Im}\, (-c - \zeta g(\zeta)) - \inf_{|\zeta|=\rho} \text{Im}\, (-c - \zeta g(\zeta)) \\ &\leq 2 \sup_{|\zeta|=\rho} |\zeta||g(\zeta)| \ \leq\ 4C_0\rho^{-1/4} + 2C_0\epsilon|w(\rho/2, -\pi)|. \end{aligned}$$

The last inequality comes from Lemma 3.1, setting $2r_1 = \rho$. Finally, integrate (2.4) to get the bound $|w|(\rho/2, -\pi) \leq |w|(\rho, -\pi)$ and choose C sufficiently large. □

Integrating u_θ around $C(\rho)$ gives $w(\rho, -\pi)$, which yields a lower bound on $\sup_{C(\rho)} u_\theta$ in terms of the separation. The oscillation bound of (3.2) then gives a lower bound for u_θ. Indeed, for ϵ sufficiently small and large ρ, u_θ is positive.

Proposition 3.3. *There exists an ϵ_0 so: Suppose Γ_u is a 3-valued ϵ-sheet on scale 1 with $\epsilon < \epsilon_0$ and $w(1, \theta) \geq C_2 > 0$. Then there exists $C_3 = C_3(C_2) \geq 2$, so that on $S^{-\pi,\pi}_{C_3,\infty}$:*

$$u_\theta(\rho, \theta) \geq \frac{C_2}{8\pi}\rho^{-\epsilon}. \tag{3.3}$$

Proof. Since $\int_{-\pi}^{\pi} u_\theta(\rho, \theta)\ d\theta = w(\rho, -\pi)$ we see $w(\rho, -\pi) \leq 2\pi \sup_{C(\rho)} u_\theta$. Using the oscillation bound (3.2) then gives the lower bound:

$$(1 - 2\pi C\epsilon)w(\rho, -\pi) - 2\pi C\rho^{-1/4} \leq 2\pi \inf_{C(\rho)} u_\theta. \tag{3.4}$$

Pick ϵ_0 so that $2\pi C\epsilon_0 \leq 1/2$. Integrating (2.4) yields $w(\rho, \theta) \geq w(1, \theta)\rho^{-\epsilon} \geq C_2\rho^{-\epsilon}$. Thus,

$$\inf_{C(\rho)} u_\theta \geq \frac{C_2}{4\pi}\rho^{-\epsilon} - C\rho^{-1/4}. \tag{3.5}$$

Since $\epsilon < 1/4$, just choose C_3 large. □

4. Decomposition of Σ

In order to decompose Σ, we use the explicit asymptotic properties found above to show that, away from the "axis," Σ consists of two strictly spiraling graphs. In particular, this implies that all intersections of Σ with planes transverse to the x_3-axis have exactly two ends. The proof of Rado's theorem then gives that $\nabla_\Sigma x_3$ is non-vanishing and so each level set consists of one unbounded smooth curve. A curvature estimate and a Harnack inequality then give the lower bound on $|\nabla_\Sigma x_3|$ near the axis. To prove Theorem 1.1 we first construct $\mathcal{R}_S$.

Lemma 4.1. *There exist constants C_1, R_1 and a sequence (y_i, s_i) of C_1 blow-up pairs of Σ so that: $x_3(y_i) < x_3(y_{i+1})$ and for $i \geq 0$, $y_{i+1} \in B_{R_1 s_i}(y_i)$ while for $i < 0$, $y_{i-1} \in B_{R_1 s_i}(y_i)$. Moreover, if $\mathcal{R}_A$ is the connected component of $\bigcup_i B_{R_1 s_i}(y_i) \cap \Sigma$ containing y_0 and $\mathcal{R}_S = \Sigma \backslash \mathcal{R}_A$, then $\mathcal{R}_S$ has exactly two unbounded components, which are (oppositely oriented) multivalued graphs u^1 and u^2 with $u^i_\theta \neq 0$. In particular, $\nabla_\Sigma x_3 \neq 0$ on the two graphs.*

Proof. Fix $\epsilon < \epsilon_0$ where ϵ_0 is given by Proposition 3.3. Using this ϵ, from Theorem 2.3 we obtain the blow-up constant C_1 and denote by C_2 the lower bound on initial separation. Suppose $0 \in \Sigma$ and that $(0,1)$ is a C_1 blow-up pair. From Theorem 2.5 there exists a constant C_{in} so that there are C_1 blow-up pairs (y_+, s_+) and (y_-, s_-) with $x_3(y_-) < 0 < x_3(y_+)$ and $y_\pm \in B_{C_{in}}$. Note by Proposition 7.3 that there is a fixed upper bound N on the number of sheets between the blow-up sheets associated to $(y_\pm, s_\pm)$ and the sheets Σ^0_i $(i = 1, 2)$ associated to $(0, 1)$.

As a consequence of Theorem 7.2, there exists an R so that all the N sheets above and the N sheets below Σ^0_i are ϵ-sheets centered on the x_3-axis on scale R. Call these pairs of 1-valued sheets Σ^j_i with $-N \leq j \leq N$. Integrating (2.4), we obtain from C_2 and N a value, C_2', so that for all Σ^j_i, the separation over ∂D_R is bounded below by C_2'. Non-vanishing of the right hand side of (3.3) is scaling invariant, so there exists a C_3 such that: on each Σ^j_i, outside of a cylinder centered on the x_3-axis of radius RC_3, $u^i_\theta \neq 0$. The chord-arc bounds of [10] (i.e. Theorem 0.5) then allow us to pick R_1 large enough so the component of $B_{R_1} \cap \Sigma$ containing 0 contains this cylinder, the points y_+, y_- and meets each Σ^j_i. Finally, we note that all the statements in the theorem are invariant under rescaling. Hence, use Theorem 2.5 to construct a sequence of C_1 blow-up pairs (y_i, s_i) satisfying the necessary conditions. □

The placement of the blow-up pairs and the strict spiraling gives:

Lemma 4.2. *For all h, there exist $\alpha, \rho_0 > 0$ so that for all $\rho > \rho_0$ the set $\Sigma \cap \{x_3 = c\} \cap \left\{x_1^2 + x_2^2 = \rho^2\right\}$ consists of exactly two points for $|c - h| \leq \alpha$.*

Proof. First note, for ρ_0 large, the intersection is never empty by the maximum principle and because Σ is proper. Without loss of generality we may assume $h = 0$ with $0 \in Z^0 = \Sigma \cap \{x_3 = 0\}$ and $|A|^2(0) \neq 0$. Let R_1 and the set of blow-up pairs be given by Lemma 4.1. There then exists ρ_0 so for $2\rho > \rho_0$, $\left\{x_1^2 + x_2^2 = \rho^2\right\} \cap Z^0$ lies in the set $\mathcal{R}_S$. If no such ρ_0 existed then, since the blow-up pairs lie outside a cone, there would exist $\delta > 0$ and a subset of the blow-up pairs (y_i, s_i) so $0 \in B_{\delta R_1 s_i}(y_i)$. However, Lemma 2.26 of [10], with $K_1 = \delta R_1$, would then imply $|A|^2(0) \leq K_2 s_i^{-2}$, or $|A|^2(0) = 0$, a contradiction. Now, for some small α and $\rho > \rho_0$, $Z^c \cap \left\{x_1^2 + x_2^2 = \rho^2\right\}$ lies in $\mathcal{R}_S$ for all $|c| < \alpha$, and so $\left\{x_1^2 + x_2^2 = \rho^2\right\} \cap$

$\{-\alpha < x_3 < \alpha\} \cap \Sigma$ consists of the union of the graphs of u^1 and u^2 over the circle ∂D_ρ, both of which are monotone increasing in height. □

As x_3 is harmonic on Σ, Proposition 1.3 is an immediate consequence of the previous result. We now show Theorem 1.1:

Proof. By Lemma 4.1 it remains to show that $|\nabla_\Sigma x_3|$ is bounded below on $\mathcal{R}_A$. Suppose that $(0, 1)$ is a blow-up pair. By the chord-arc bounds of [10], there exists γ large enough so that the intrinsic ball of radius γR_1 contains $\Sigma \cap B_{R_1}$. Lemma 2.26 of [10] implies that curvature is bounded in $B_{2\gamma R_1} \cap \Sigma$ uniformly by K. The function $v = -2 \log |\nabla_\Sigma x_3| \geq 0$ is well defined and smooth by Proposition 1.3 and standard computations give $\Delta_\Sigma v = |A|^2$. Then, since $|\nabla_\Sigma x_3| = 1$ somewhere in the component of $B_1(0) \cap \Sigma$ containing 0, we can apply a Harnack inequality (see Theorems 9.20 and 9.22 in [12]) to obtain an upper bound for v on the intrinsic ball of radius γR_1 that depends only on K. Consequently, there is a lower bound ϵ_0 on $|\nabla_\Sigma x_3|$ in $\Sigma \cap B_{R_1}$. Since this bound is scaling invariant, the same bound holds around any blow-up pair. Finally, any bounded component, Ω, of $\mathcal{R}_S$ has boundary in $\mathcal{R}_A$ and so, since v is subharmonic, $|\nabla_\Sigma x_3| \geq \epsilon_0$ on Ω. Thus, by adjoining all such bounded Ω to $\mathcal{R}_A$ we obtain Theorem 1.1. □

5. Concluding Uniqueness

Since $\nabla_\Sigma x_3$ is non-vanishing and the level sets of x_3 in Σ consist of a single curve, the map $z = x_3 + i x_3^* : \Sigma \to \mathbb{C}$ is a global holomorphic coordinate (here x_3^* is the harmonic conjugate of x_3). Additionally, $\nabla_\Sigma x_3 \neq 0$ implies that the normal of Σ avoids $(0, 0, \pm 1)$. Thus, the stereographic projection of the Gauss map, denoted by g, is a holomorphic map $g : \Sigma \to \mathbb{C} \setminus \{0\}$. By monodromy, there exists a holomorphic map $h = h_1 + i h_2 : \Sigma \to \mathbb{C}$ so that $g = e^h$. We will use h to show that z is actually a conformal diffeomorphism between Σ and $\mathbb{C}$. As the same is then true for h, embeddedness and the Weierstrass representation implies Σ is the helicoid.

5.1. **Structure of h.** We note the following relation between $\nabla_\Sigma x_3$, g and h:

$$|\nabla_\Sigma x_3| = 2 \frac{|g|}{1 + |g|^2} \leq 2 e^{-|h_1|}. \tag{5.1}$$

An immediate consequence of (5.1) and the decomposition of Theorem 1.1 is that there exists $\gamma_0 > 0$ so on $\mathcal{R}_A$, $|h_1(z)| \leq \gamma_0$. This imposes strong rigidity on h:

Proposition 5.1. *Let $\Omega_\pm = \{x \in \Sigma : \pm h_1(x) \geq 2\gamma_0\}$ then h is a proper conformal diffeomorphism from $\Omega_\pm$ onto the closed half-spaces $\{z : \pm \mathrm{Re}\, z \geq 2\gamma_0\}$.*

Proof. Let $\gamma > \gamma_0$ be a a regular value of h_1. Such γ exists by Sard's theorem and indeed form a dense subset of (γ_0, ∞). We first claim that the smooth submanifold $Z = h_1^{-1}(\gamma)$ has a finite number of components. Note that Z is non-empty by (2.5) and (5.1). By construction, Z is a subset of $\mathcal{R}_S$ and, up to choosing an orientation, Z lies in the graph of u^1, which we will henceforth denote as u. Let us parameterize one of the components of Z by $\phi(t)$, non-compact by the maximum principle, and write $\phi(t) = \Phi_u(\rho(t), \theta(t))$.

At the point $\Phi_u(\rho, \theta)$ we compute:

$$g(\rho, \theta) = -\frac{1}{\sqrt{1 + |\nabla u|^2} - 1} \left(u_\rho(\rho, \theta) + i \frac{u_\theta(\rho, \theta)}{\rho} \right) e^{i\theta}. \tag{5.2}$$

Since $u_\theta(\rho(t),\theta(t))>0$, there exists a function $\tilde{\theta}(t)$ with $\pi<\tilde{\theta}(t)<2\pi$ such that

$$|\nabla u|(\rho(t),\theta(t))e^{i\tilde{\theta}(t)}=-u_\rho(\rho(t),\theta(t))-i\frac{u_\theta(\rho(t),\theta(t))}{\rho(t)}. \tag{5.3}$$

Thus $h_2(\phi(t))=\theta(t)+\tilde{\theta}(t)$.

We now claim $\lim_{t\to\pm\infty}|h_2(\phi(t))|=\infty$. Suppose $\lim_{t\to\infty}h_2(\phi(t))=R<\infty$ and $h_2(\phi(t))<R$. Then, the formula for $h_2(\phi(t))$ implies that for t large $\phi(t)$ lies in one sheet. The decay estimates (2.5) together with (5.1) imply $\rho(t)$ cannot became arbitrarily large and so the positive end of ϕ lies in a compact set. Thus, there is a sequence of points $p_j=\phi(t_j)$, with t_j monotonically increasing to ∞, so $p_j\to p_\infty\in\Sigma$. By the continuity of h_1, $p_\infty\in Z$, and since $h_2(p_j)$ is monotone increasing with supremum R, $h_2(p_\infty)=R$, and so p_∞ is not in ϕ. However, $p_\infty\in Z$ implies $h'(p_\infty)\neq 0$ and so h restricted to a small neighborhood of p_∞ is a diffeomorphism onto its image, contradicting ϕ coming arbitrarly close to p_∞.

Thus, the formula for $h_2(\phi(t))$ and the bound on $\tilde{\theta}$ show that $\theta(t)$ must extend from $-\infty$ to ∞. We now conclude that there are at most a finite number of components of Z. Namely, since $\theta(t)$ runs from $-\infty$ to ∞ we see that every component of Z must meet the curve $\eta(\rho)=\Phi_u(\rho,0)\in\mathcal{R}_S$. Again, the gradient decay of (2.5) says that the set of intersections of Z with η lies in a compact set, and so consists of a finite number of points. Now, suppose there was more than one component of Z. Looking at the intersection of Z with η, we order these components innermost to outermost; parameterize the innermost curve by $\phi_1(t)$ and the outermost by $\phi_2(t)$. Pick τ a regular value for h_2, and parameterize the component of $h_2^{-1}(\tau)$ that meets ϕ_1 by $\sigma(t)$, writing $\sigma(t)=\Phi_u(\rho(t),\theta(t))$ in $\mathcal{R}_S$. From the formula for h_2, $|\theta(t)-\tau|\leq 2\pi$. Again, $\sigma(t)$ cannot have an end in a compact set, so $\rho(t)\to\infty$. Hence, σ must also intersect ϕ_2 contradicting the monotonicity of h_1 on σ.

Hence, when $\gamma>\gamma_0$, is a regular value of h_1, $h_1^{-1}(\gamma)$ is a single smooth curve. We claim this implies that all $\gamma>\gamma_0$ are regular values. Suppose $\gamma'>\gamma_0$ were a critical value of h_1. Then, as h_1 is harmonic, the proof of Rado's theorem implies for $\gamma>\gamma_0$, a regular value of h_1 near γ', $h_1^{-1}(\gamma)$ would have at least two components. Thus, $h:\Omega_+\to\{z:\operatorname{Re} z\geq 2\gamma_0\}$ is a conformal diffeomorphism that maps boundaries onto boundaries, immediately implying h is also proper on Ω_+, and similarly for Ω_-. □

By looking at z, which already has well understood behavior away from ∞, we see that Σ is conformal to $\mathbb{C}$ with z providing an identification.

Proposition 5.2. *The map $h\circ z^{-1}:\mathbb{C}\to\mathbb{C}$ is linear.*

Proof. We first show that z is a conformal diffeomorphism between Σ and $\mathbb{C}$, that is z is onto. This follows if we show x_3^* goes from $-\infty$ to ∞ on the level sets of x_3. The key fact is: each level set of x_3 has one end in Ω_+ and the other in Ω_-. This is an immediate consequence of the radial gradient decay on level sets of x_3 forced by the one-sided curvature estimate. Indeed, x_3 runs from $-\infty$ to ∞ along the curve $\partial\Omega_+$ and so $z(\partial\Omega_+)$ splits $\mathbb{C}$ into two components with only one, V, meeting $z(\Omega_+)=U$. After conformally straightening the boundary of V (using the Riemann mapping theorem) and precomposing with $h|_{\Omega_+}^{-1}$ we obtain a map from a closed half-space into a closed half-space with the boundary mapped to the boundary. We claim that this map is neccesarily onto, that is U equals $\bar{V}$. Suppose it was not onto, then a Schwarz reflection would give a holomorphic map from $\mathbb{C}$ into a simply connected proper subset of $\mathbb{C}$. Because the latter is conformally a disk, Liouville's theorem would imply this map was constant, a contradiction. As a consequence, if $p \to \infty$ in Ω_+ then $z(p) \to \infty$, with the same true in Ω_-. Thus, along each level set of x_3, $|x_3^*(p)| \to \infty$ and so z is onto. Then, by the level set analysis in the proof of 5.1 and Picard's theorem, $h \circ z^{-1}$ is a polynomial and is indeed linear. $\square$

5.2. **Concluding Uniqueness.** After a translation in $\mathbb{R}^3$ and a rebasing of x_3^*, $h(z) = \alpha z$ for some $\alpha \in \mathbb{C}$. As dz is the height differential, the Weierstrass representation gives $x_1(it) = |\alpha|^{-2}(\alpha_2 \sinh(\alpha_2 t)\sin(\alpha_1 t) - \alpha_1 \cosh(\alpha_2 t)\cos(\alpha_1 t))$ and $x_2(it) = |\alpha|^{-2}(\alpha_2 \sinh(\alpha_2 t)\cos(\alpha_1 t) + \alpha_1 \cosh(\alpha_2 t)\sin(\alpha_1 t))$ where $\alpha = \alpha_1 + i\alpha_2$. By inspection this curve is only embedded when $\alpha_1 = 0$, i.e. if $\alpha = i\alpha_2$. The factor α_2 corresponds to a homothetic rescaling and so Σ is the helicoid.

6. Local Result

Consider two surfaces $\Sigma_1, \Sigma_2 \subset \mathbb{R}^3$, so that Σ_2 is the graph of ν over Σ_1. Then the map $\phi : \Sigma_1 \to \Sigma_2$ defined as $\phi(x) = x + \nu(x)\mathbf{n}(x)$ is smooth. Moreover, if ν is small in a C^1 sense, ϕ is an "almost isometry".

Lemma 6.1. *Let Σ_2 be the graph of ν over Σ_1, with $\Sigma_1 \subset B_R$, $\partial\Sigma_1 \subset \partial B_R$ and $|A_{\Sigma_1}| \leq 1$. Then, for ϵ sufficiently small, $|\nu| + |\nabla_{\Sigma_1}\nu| \leq \epsilon$ implies ϕ is a diffeomorphism with $1 - \epsilon \leq ||d\phi|| \leq 1 + \epsilon$.*

Proof. For ϵ sufficiently small (depending on Σ_1), ϕ is injective. Working in $\mathbb{R}^3$, given orthonormal vectors $e_1, e_2 \in T_p\Sigma_1$ we compute:

$$d\phi_p(e_i) = e_i + \langle \nabla_{\Sigma_1}\nu(p), e_i\rangle \mathbf{n}(p) + \nu(p) D\mathbf{n}_p(e_i). \tag{6.1}$$

The last two terms are together controlled by ϵ. Hence, $1-\epsilon < |d\phi_p(e_i)| < 1+\epsilon$. $\square$

Proof. (of Theorem 1.5) By rescaling we may assume that $s = 1$. We proceed by contradiction. Suppose no such R' existed for fixed ϵ, R. That is, there exists a sequence of counter-examples; PEMDs Σ_i' with $\Sigma_i' \subset B_{R_i}$, $\partial\Sigma_i' \subset \partial B_{R_i}$, $(0,1)$ a C blow-up pair of each Σ_i' and $R \leq R_i \to \infty$, but Σ_i, the component of $B_R \cap \Sigma_i'$ containing zero, not close to a helicoid.

By definition, $|A_{\Sigma_i'}(0)|^2 = C > 0$ for all Σ_i' and so the lamination theory of Colding and Minicozzi implies that a subsequence of the Σ_i' converge smoothly and with multiplicity one to Σ_∞, a complete embedded minimal disk. Namely, in any ball centered at 0 the curvature of Σ_i is uniformly bounded by Lemma 2.26 of [10]. Furthermore, the chord-arc bounds of [10] give uniform area bounds and so by standard compactness arguments one has smooth convergence (possibly with multiplicity) to Σ_∞. If the multiplicity of the convergence is greater than 1, then one can construct a positive solution to the Jacobi equation (see Appendix B of [2]). That implies Σ_∞ is stable, and thus a plane by the Bernstein theorem, contradicting the curvature at 0. As Corollary 0.7 of [10] gives properness of Σ_∞, Theorem 1.4 implies Σ_∞ is a helicoid. We may, by rescaling, assume Σ_∞ has curvature 1 along the axis.

For any fixed R' a subsequence of $\Sigma_i' \cap B_{R'}$ converges to $\Sigma_\infty \cap B_{R'}$ in the smooth topology. And so, for any ϵ, with i sufficiently large, we find a smooth ν_i defined on a subset of Σ_∞ so that $|\nu_i| + |\nabla_{\Sigma_\infty}\nu_i| < \epsilon$ and the graph of ν_i is $\Sigma_i' \cap B_{R'}$. Choosing R' large enough to ensure minimizing geodesics between points in Σ_i lie in $\Sigma_i' \cap B_{R'}$ (using the chord-arc bounds of [10]), Lemma 6.1 gives the desired contradiction. $\square$

7. Appendix

7.1. **Structural Results of Colding and Minicozzi.** To show that near a blow-up pair there is a single N-valued ϵ-sheet, one needs two results of Colding and Minicozzi. First, from [7], is the existence, near a blow-up point, of N-valued graphs that extend almost to the boundary. Then, by [6], since a large number of sheets gives (2.4), after a suitable rotation one has an ϵ-sheet.

Theorem 7.1. *Given $\epsilon > 0$, $N \in \mathbb{Z}^+$, there exist $C_1, C_2 > 0$ so: Suppose that $(0,s)$ is a C_1 blow-up pair of Σ. Then there exists (after a rotation of $\mathbb{R}^3$) an N-valued ϵ-sheet $\Sigma_0 = \Gamma_{u_0}$ on the scale s. Moreover, the separation over ∂D_s of Σ_0 is bounded below by $C_2 s$.*

Proof. Proposition II.2.12 of [6] and standard elliptic estimates give an $N_\epsilon \in \mathbb{Z}^+$ and $\delta_\epsilon > 0$ so that if u satisfies (2.3) on $S^{-\pi N_\epsilon, \pi N_\epsilon}_{e^{-N_\epsilon}, \infty}$ and $\Gamma_u \subset \mathbf{C}_{\delta_\epsilon}$, then on $S^{0,2\pi}_{1,\infty}$ we have all the terms of (2.4) bounded (by $\epsilon/2$) except $|\nabla u|$. Setting $\tau = \min\left\{\frac{\epsilon}{4}, \frac{\delta_\epsilon}{2}\right\}$ and $N_0 = N + N_\epsilon + 2$, apply Corollary 4.14 from [7] to obtain C. That is, if $(0,t)$ is a C blow-up pair, then the corollary gives an N_0-valued graph u defined on $S^{-\pi N_0, \pi N_0}_{t,\infty}$ with $\Gamma_u \subset \mathbf{C}_\tau \cap \Sigma$ and $|\nabla u| \leq \tau$. Hence by above (and a rescaling) we see that u satisfies (2.4) on $S^{-\pi N, \pi N}_{e^{N_\epsilon} t, \infty}$. At this point we do not a priori know that $\lim_{\rho \to \infty} \nabla u(\rho, 0) = 0$. However, there is an asymptotic tangent plane. Thus after a small rotation to make this parallel to the x_1-x_2 plane (and a small adjustment to τ and t), we may assume the limit is zero.

Proposition 4.15 of [7] gives a $\beta > 0$ so that $w(t,\theta) \geq \beta t$. Integrating (2.4), we obtain from this a C_2 so that $w(e^{N_\epsilon} t, \theta) \geq C_2 e^{N_\epsilon} t$. Finally, if we set $C_1 = C e^{N_\epsilon}$ then $(0,s)$ being a C_1 blow-up pair implies that $(0, e^{-N_\epsilon} s)$ is a C blow-up pair. This gives the result. □

Once we have a single sheet, we can immediately apply the one-sided curvature estimate to obtain a graphical region inside of a cone which, moreover, satisfies (2.4) in a smaller cone. Results along these lines can by found in [3] and [8]. We will need:

Theorem 7.2. *Suppose Σ contains a 4-valued ϵ-sheet Σ_0 on the scale 1 with ϵ sufficiently small. Then there exist $R \geq 1$, $\delta > 0$ depending only on ϵ such that the component of $\Sigma \cap (\mathbf{C}_\delta \backslash B_R)$ that contains the 3-valued middle sheet on scale R of Σ_0 can be expressed as the multivalued graph of a function, u, which satisfies* (2.4).

7.2. **Blow-up Pairs.** By the lamination theory, the existence of a blow-up pair imposes strong control on nearby geometry. The chord-arc bounds and Lemma 2.26 of [10] are examples. We also have:

Proposition 7.3. *Given K, there is an N so that: If (y_1, s_1) and (y_2, s_2) are C blow-up pairs of Σ with $y_2 \in B_{K s_1}(y_1)$, then the number of sheets between the associated blow-up sheets is at most N.*

Proof. We note that for a large, universal constant C' the area of $B_{C' K s_1}(y_1) \cap \Sigma$ gives a bound on N, so it is enough to uniformly bound this area. The chord-arc bounds of [10] give a uniform constant γ depending only on C' so that $B_{C' K s_1}(y_1) \cap \Sigma$ is contained in $\mathcal{B}_{\gamma K s_1}(y_1)$ the intrinsic ball in Σ of radius $\gamma K s_1$. Furthermore, Lemma 2.26 of [10] gives a uniform bound on the curvature of Σ in $\mathcal{B}_{\gamma K s_1}(y_1)$ and hence a uniform bound on the area of $\mathcal{B}_{\gamma K s_1}(y_1)$. Since $B_{C' K s_1}(y_1) \cap \Sigma \subset \mathcal{B}_{\gamma K s_1}(y_1)$ it also has uniformly bounded area. □

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