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Distortions of the helicoid

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Abstract Colding and Minicozzi have shown that an embedded minimal disk $0 \in \Sigma \subset B_R$ in $\mathbb{R}^3$ with large curvature at 0 looks like a helicoid on the scale of R. Near 0, this can be sharpened: on the scale of $|A|^{-1}(0)$, Σ is close, in a Lipschitz sense, to a piece of a helicoid. We use surfaces constructed by Colding and Minicozzi to see this description cannot hold on the scale R.

Keywords Differential geometry · Minimal surfaces

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In [4–7], Colding and Minicozzi give a complete description of the structure of embedded minimal disks in a ball in $\mathbb{R}^3$. Roughly speaking, they show that any such surface is either modeled on a plane (i.e. nearly graphical) or is modeled on a helicoid (i.e. two multi-valued graphs glued together along an axis). In the latter case, the distortion may be quite large. For instance, in [9], Meeks and Weber "bend" the helicoid; that is, they construct minimal surfaces where the axis is an arbitrary $C^{1,1}$ curve (see Fig. 1). A more serious example of distortion is given by Colding and Minicozzi in [3]. There they construct a sequence of minimal disks modeled on the helicoid, but where the ratio between the scales (a measure of the tightness of the spiraling of the multi-graphs) at different points of the axis becomes arbitrarily large (see Fig. 2). Note, locally, near points of large curvature, the surface is close to a helicoid, and so the distortions are necessarily global in nature.

Following [5] we make the meaning of large curvature precise by saying a pair $(y, s) \in \Sigma \times \mathbb{R}^+$ is a (C) *blow-up pair* if $\sup_{B_s(y)\cap\Sigma} |A_\Sigma|^2 \leq 4C^2 s^{-2} = 4|A_\Sigma|^2(y)$ (C large and

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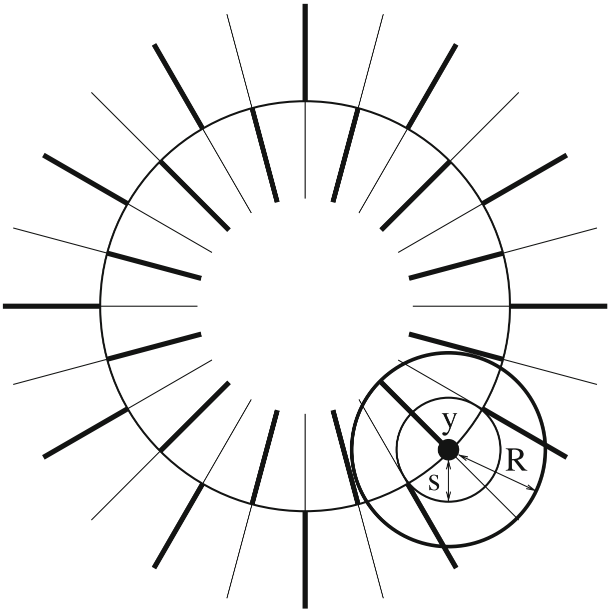

Fig. 1 A cross section of one of Meeks and Weber's examples, with the axis as a circle. We indicate a subset which is a disk. Here R is the outer scale of said disk and s the blow-up scale

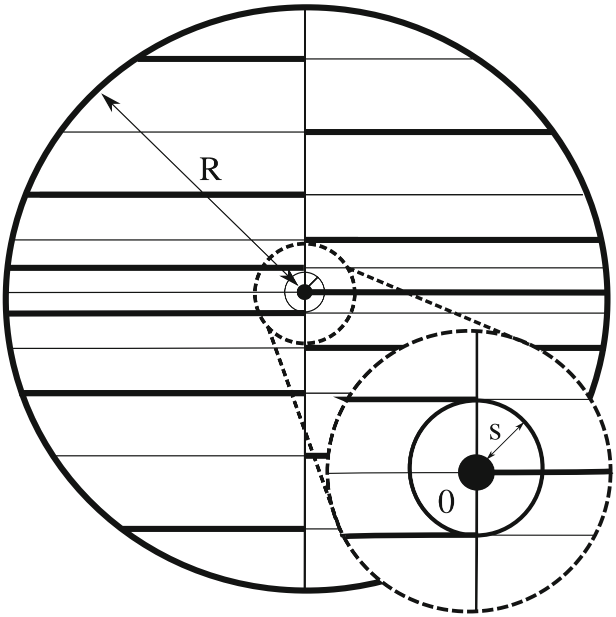

Fig. 2 A cross section of one of Colding and Minicozzi's examples. We indicate the two important scales: $R = 1$ the outer scale and s the blow-up scale. (Here $(0, s)$ is a blow-up pair.)

fixed and $\Sigma \subset \mathbb{R}^3$ minimal), here $B_s(y)$ is a ball of radius s centered at y in $\mathbb{R}^3$ and $|A_\Sigma|$ is the norm of the second fundamental form of Σ. For Σ minimal with $\partial\Sigma \subset \partial B_R = \partial B_R(0)$ where $(0, s)$ is a blow-up pair, there are two important scales; R the outer scale and s the blow-up scale. The work of Colding and Minicozzi gives a value $0 < \Omega < 1$ so that the component of $\Sigma \cap B_{\Omega R}$ containing 0 consists of two multi-valued graphs glued together (see for instance Lemma 2.5 of [8] for a self-contained explanation). On the other hand, Theorem 1.5 of [1] shows that on the scale of s (provided R/s is large), Σ is bi-Lipschitz to a piece of a helicoid with Lipschitz constant near 1. Using the surfaces constructed in [3] (which are the most distorted currently known) we show that such a result cannot hold on the outer scale and indeed fails to hold on certain smaller scales:

Theorem 0.1 *Given* $1 > \Omega, \epsilon > 0$ *and* $1/2 > \gamma \geq 0$ *there exists an embedded minimal disk* $0 \in \Sigma$ *with* $\partial\Sigma \subset \partial B_R$ *and* $(0, s)$ *a blow-up pair so: the component of* $B_{\Omega R^{1-\gamma}s^\gamma} \cap \Sigma$ *containing* 0 *is not bi-Lipschitz to a piece of a helicoid with Lipschitz constant in* $((1+\epsilon)^{-1}, 1+\epsilon)$.

Recall that as an application of their work on the structure of disks, Colding and Minicozzi proved a compactness result for sequences of embedded minimal disks $0 \in \Sigma_i \subset \mathbb{R}^3$ as long as $\partial\Sigma_i \subset \partial B_{R_i}$ and $R_i \to \infty$. In particular, they show there are only two options. Either such a sequence contains a sub-sequence converging smoothly on compact sets to a complete embedded minimal disk or, if the curvature is unbounded in some compact subset of $\mathbb{R}^3$, the convergence is (in a certain sense, see [7] for details) to a singular minimal lamination of parallel planes. The surfaces constructed by Colding and Minicozzi in [3] show that the condition that the boundaries of the surface go to infinity is essential, i.e this compactness result is global in nature. In a similar vein, the result depends very strongly on the ambient geometry of the three-manifold. In particular, in the proof of their compactness result, Colding and Minicozzi rely heavily on a flux argument (the details of which are in [2]). That is they use that the coordinate functions of $\mathbb{R}^3$ restrict to harmonic functions on minimal $\Sigma \subset \mathbb{R}^3$, a fact that generalizes only to certain other highly symmetric three-manifolds.

One of the most important problems in this area is determining when a Colding and Minicozzi type of compactness result (or indeed any compactness result) extends to surfaces embedded in more general three-manifolds. Understanding precisely the best scale for which the Lipschitz approximation holds (for which Theorem 0.1 gives an upper bound) may be an important tool to establish removable singularities theorems for minimal laminations in arbitrary Riemannian manifolds. In turn, such results could prove key to proving more general compactness theorems.

To produce our example, we first recall the surfaces constructed in [3]:

Theorem 0.2 (Theorem 1 of [3]) *There is a sequence of compact embedded minimal disks* $0 \in \Sigma_i \subset B_1 \subset \mathbb{R}^3$ *with* $\partial\Sigma_i \subset \partial B_1$ *containing the vertical segment* $\{(0, 0, t) : |t| \leq 1\} \subset \Sigma_i$ *such that the following conditions are satisfied:*

(1) $\lim_{i\to\infty} |A_{\Sigma_i}|^2(0) \to \infty$
(2) $\sup_{\Sigma_i} |A_{\Sigma_i}|^2 \leq 4|A_{\Sigma_i}|^2(0) = 8a_i^{-4}$ *for a sequence* $a_i \to 0$
(3) $\sup_i \sup_{\Sigma_i \setminus B_\delta} |A_{\Sigma_i}|^2 < K\delta^{-4}$ *for all* $1 > \delta > 0$ *and* K *a universal constant.*
(4) $\Sigma_i \setminus \{x_3 - axis\} = \Sigma_{1,i} \cup \Sigma_{2,i}$ *for multi-valued graphs* $\Sigma_{1,i}$ *and* $\Sigma_{2,i}$.

Remark 0.3 (2) and (3) are slightly sharper than what is stated in Theorem 1 of [3], but follow easily. (2) follows from the Weierstrass data (see Eq. 2.3 of [3]). This also gives (3) near the axis, whereas away from the axis use (4) and Heinz's curvature estimates.

Next introduce some notation. For a surface Σ (with a smooth metric) we denote intrinsic balls of radius s, centered at p, by $\mathcal{B}_s^\Sigma(p)$ and define the (*intrinsic*) *density ratio* at a point p as: $\theta_s(p,\Sigma) = (\pi s^2)^{-1}\text{Area}(\mathcal{B}_s^\Sigma(p))$. When Σ is immersed in $\mathbb{R}^3$ and has the induced metric, $\theta_s(p,\Sigma) \leq \Theta_s(p,\Sigma) = (\pi s^2)^{-1}\text{Area}(B_s(p)\cap\Sigma)$, the usual (extrinsic) density ratio. Importantly, the intrinsic density ratio is well-behaved under bi-Lipschitz maps. Indeed, if $f : \Sigma \to \Sigma'$ is injective and with $\alpha^{-1} < \text{Lip} f < \alpha$ (where $\text{Lip} f$ is the Lipschitz constant of f), then:

$$\alpha^{-4}\theta_{\alpha^{-1}s}(p,\Sigma) \leq \theta_s(f(p),\Sigma') \leq \alpha^4\theta_{\alpha s}(p,\Sigma). \tag{0.1}$$

This follows from the inclusion, $\mathcal{B}^\Sigma_{\alpha^{-1}s}(f^{-1}(p)) \subset f^{-1}(\mathcal{B}_s^{\Sigma'}(p))$ and the behavior of area under Lipschitz maps, $\text{Area}(f^{-1}(\mathcal{B}_s^{\Sigma'}(p)) \leq (\text{Lip} f^{-1})^2\text{Area}(\mathcal{B}_s^{\Sigma'}(p))$.

Note that by standard area estimates for minimal graphs, if $\Sigma\cap B_s(p)$ is a minimal graph then $\theta_s(p,\Sigma)\leq 2$. In contrast, for a point near the axis of a helicoid, for large s the density ratio is large. Thus, in a helicoid the density ratio for a fixed, large s measures, in a rough sense, the distance to the axis. More generally, this holds near blow-up pairs of embedded minimal disks:

Lemma 0.4 *Given $D > 0$ there exists $R > 1$ so: If $0 \in \Sigma \subset B_{2Rs}$ is an embedded minimal disk with $\partial\Sigma \subset \partial B_{2Rs}$ and $(0,s)$ a blow-up pair then $\theta_{Rs}(0,\Sigma) \geq D$.*

Proof We proceed by contradiction, that is suppose there were a $D > 0$ and embedded minimal disks $0 \in \Sigma_i$ with $\partial\Sigma_i \subset \partial B_{2R_is}$ with $R_i \to \infty$ and $(0,s)$ a blow-up pair so that $\theta_{R_is}(0,\Sigma_i) \leq D$. The chord-arc bounds of [8] imply there is a $1 > \gamma > 0$ so $\mathcal{B}^{\Sigma_i}_{R_is}(0) \supset \Sigma_i \cap B_{\gamma R_is}$. Hence, the intrinsic density ratio bounds the extrinsic density ratio, i.e. $D \geq \theta_{R_is}(p,\Sigma_i) \geq \gamma^2\Theta_{\gamma R_is}(p,\Sigma_i)$. Then, by a result of Schoen and Simon [10] there is a constant $K = K(D\gamma^{-2})$, so $|A_{\Sigma_i}|^2(0) \leq K(\gamma R_is)^{-2}$. For R_i large this contradicts that $(0,s)$ is a blow-up pair for all Σ_i. □

Remark 0.5 Note that the above does not depend on the strength of chord-arc bounds. In fact, it is also an immediate consequence of the fact that intrinsic area bounds on a disk give total curvature bounds. In turn, the total curvature bounds again yield uniform curvature bounds. See Sect. 1 of [5] for more detail.

In order to show the existence of the surface Σ of Theorem 0.1, we exploit the fact that two points on a helicoid that are equally far from the axis must have the same density ratio. Assuming the existence of a Lipschitz map between Σ and a helicoid, we get a contradiction by comparing the densities for two appropriately chosen points that map to points equally far from the axis of the helicoid.

Proof (of Theorem 0.1) Fix $1 > \Omega, \epsilon > 0$ and $1/2 > \gamma \geq 0$ and set $\alpha = 1+\epsilon$. Let Σ_i be the surfaces of Theorem 0.2; we claim for i large, Σ_i will be the desired example. Suppose this was not the case. Setting $s_i = Ca_i^2/\sqrt{2}$, where a_i is as in (2) and C is the blow-up constant, one has $(0,s_i)$ is a blow-up pair in Σ_i, since $\sup_{\Sigma_i\cap B_{s_i}}|A_{\Sigma_i}|^2 \leq 8a_i^{-4} = 4C^2s_i^{-2} = 4|A_{\Sigma_i}|^2(0)$, moreover, $s_i \to 0$. Hence, with $R_i = \Omega s_i^\gamma < 1$, the component of $B_{R_i}\cap\Sigma_i$ containing 0, Σ_i', is bi-Lipschitz to a piece of a helicoid with Lipschitz constant in (α^{-1},α). That is, there are subsets Γ_i of helicoids and diffeomorphisms $f_i : \Sigma_i' \to \Gamma_i$ with $\text{Lip} f_i \in (\alpha^{-1},\alpha)$.

We now begin the density comparison. First, Lemma 0.4 implies there is a constant $r > 0$ so for i large $\theta_{rs_i}(0,\Sigma_i') \geq 4\alpha^8$ and thus by (0.1) $\theta_{\alpha rs_i}(f_i(0),\Gamma_i) \geq 4\alpha^4$. We proceed to find a point with small density on Σ_i that maps to a point on Γ_i equally far from the axis as $f_i(0)$ (which has large density).

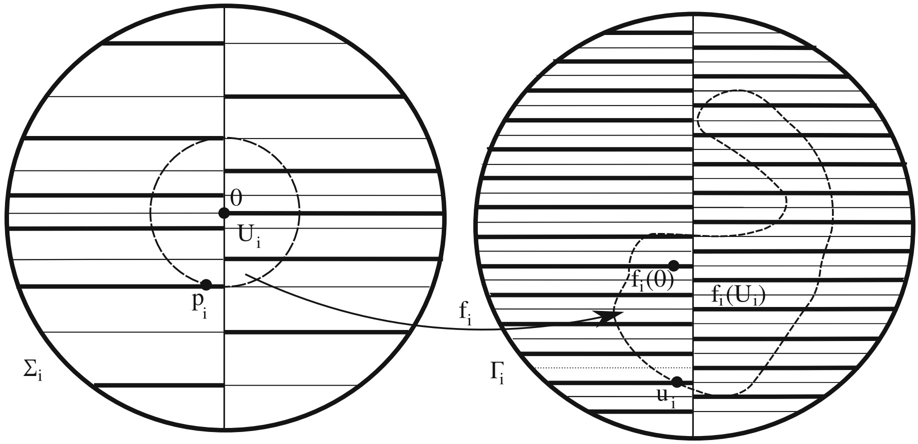

Fig. 3 The points p_i and u_i. Note that the density ratio of u_i is much larger than the density ratio of p_i

Let U_i be the (interior) of the component of $B_{1/2R_i} \cap \Sigma_i$ containing 0. Note for i large enough, as $s_i/R_i \to 0$, the distance between ∂U_i and $\partial \Sigma_i{}'$ is greater than $4\alpha^2 r s_i$. Similarly, for $p \in \partial U_i$ for i large, $p' \in \mathcal{B}^{\Sigma_i'}_{4\alpha^2 r s_i}(p)$ implies $|p'| \geq \frac{1}{4} R_i$. Hence, property (3) gives that $|A_{\Sigma_i'}|^2(p') \leq K' s_i^{-4\gamma}$. Thus, for i sufficiently large $\mathcal{B}_{\alpha^2 r s_i}(p)$ is a graph and so $\theta_{\alpha^2 r s_i}(p, \Sigma_i') \leq 2$. Pick $u_i \in \partial f(U_i)$ at the same distance to the axis as $f_i(0)$ and so the density ratio is the same at both points (see Fig. 3). As $f_i(U_i)$ is an open subset of Γ_i containing $f_i(0)$, $p_i = f_i^{-1}(u_i) \in \partial U_i$. Notice that $\theta_{\alpha r s_i}(u_i, \Gamma_i) = \theta_{\alpha r s_i}(f_i(0), \Gamma_i) \geq 4\alpha^4$ so $2\alpha^4 \geq \alpha^4 \theta_{\alpha^2 r s_i}(p_i, \Sigma_i') \geq 4\alpha^4$. □

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