Supplementary Problems 3

(1) (Exercise 5.4.15) Let a, b, c be distinct integers. Can the polynomial $(x-a)(x-b)(x-c)-1$ be factored into the product of two polynomials with integer coefficients?

(2) (Exercise 5.4.17) Let $f(x) = x^n + 5x^{n-1} + 3$ where $n > 1$ is an integer. Prove that $f(x)$ cannot be expressed as the product of two polynomials, each of which has all its coefficients integers and degree at least 1. (Hint: Mimic the proof of Eisenstein's criterion.)

(3) Suppose that $x, y, z > 0$ and $x + y + z = 1$. Prove that

$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right) \geq 64.$$

(4) Prove the *weighted* AM-GM inequality; if $x_1, \ldots, x_n > 0$ and $w_1, \ldots, w_n > 0$ with $w_1 + \cdots + w_n = 1$, then

$$w_1 x_1 + \cdots + w_n x_n \geq x_1^{w_1} \cdots x_n^{w_n}.$$

When does the equality hold? You may assume that all w_i are *rational* numbers. If you took Analysis (110.405), you should be able to deduce the general case from this.

(5) Let E be the ellipsoid given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

for some $a, b, c > 0$. Find the volume in terms of a, b and c of the largest rectangular box that can fit inside E with faces parallel to the coordinate planes. (No calculus please.)