Supplementary Problems 2

(1) Prove that, given any 5 points chosen in an equilateral triangle of side-length 1, there are two points whose distance apart is at most 1/2. Repeat the question with "5 points" replaced by "10 points" and "at most 1/2" by "at most 1/3". In general, by "$n^2 + 1$ points" and by "at most $1/n$".

(2) Suppose that we have 27 *distinct* positive odd numbers, all less than 100. Show that there is a pair of numbers whose sum is 102.

(3) (Exercise 3.3.18) Choose any $(n + 1)$-elements subset from $\{1, 2, \ldots, 2n\}$. Show that this subset must contain two integers that are relatively prime.

(4) (Exercise 3.3.19) People are seated around a circular table at a restaurant. The food is placed on a circular platform in the center of the table, and this circular platform can rotate. Each person ordered a different entrée, and it turns out that no one has the correct entrée in front of him or her. Show that it is possible to rotate the platform so that at least two people will have the correct entrée.

(5) How many trees can farmer Fred plant on his 100 m square field (that is, a field whose shape is a square with side-length 100 m) if they are to be no closer than 10 m apart? Neglect the thickness of the trees.

(6) Let n and k be positive integers so that $n^k > (k + 1)!$. Define

$$\mathcal{M} = \{(x_1, x_2, \ldots, x_k) \mid x_i \text{ is an integer with } 1 \leq x_i \leq n \text{ for all } i\}.$$

Prove that if $A \subseteq \mathcal{M}$ has $(k+1)!+1$ elements, then there are two elements $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ in A so that $(k + 1)!$ divides $(a_1 - b_1)(a_2 - b_2) \cdots (a_k - b_k)$.