Supplementary Problems 1

(1) Let $f(x, y)$ be a polynomial, which is symmetric in x and y, that is, $f(x, y) = f(y, x)$. Put $s = x + y$ and $t = xy$. Prove that there exists a polynomial $g(s, t)$ in s and t such that $g(s, t) = f(x, y)$.

(2) For positive integers n and k, define

$$F(n, k) = \sum_{r=1}^{n} r^{2k-1}.$$

Prove that $F(n, 1)$ divides $F(n, k)$.

(3) Imagine a unit square on the xy-plane, whose vertices are $(0, 0), (1, 0), (1, 1)$ and $(0, 1)$. A particle P at $(1, 1)$ is shot toward the x-axis (not necessarily vertically) within the square. After it hits the point $(a, 0)$ on the x-axis for some a with $0 \leq a \leq 1$, it bounces back with the rule of *incidence angle = reflection angle*. And, it keeps bouncing on each of the four sides of the square in the same way. Determine all the values of a such that P eventually hits one of the four vertices.

(4) Given a polygon P, a subset of the set of edges of P is *non-contiguous* if no two edges in the set share a common vertex. Show that, if $f(n)$ denotes the number of non-contiguous subsets of an n-gon, then $f(4n) + 2$ is a perfect square for all integers $n \geq 1$.

(5) Suppose that S is a set of real numbers such that $0 \in S$ and, for all $x \in S$, $\sin x \in S$ and $\cos x \in S$. Show that for all $0 \leq a < b \leq 1$ there exists $x \in S$ such that $a \leq x \leq b$.