FINAL EXAM - SOLUTION

PUTNAM PROBLEM SOLVING

(1) Let d_1, d_2, d_3 be the integers with
$$b - c = d_1(a - b) \quad c - a = d_2(b - c) \quad a - b = d_3(c - a).$$
Then, it is easy to show that $d_1 d_2 d_3 = 1$. Hence, either $d_1 = d_2 = d_3 = 1$ or, say, $d_1 = d_2 = -1$ and $d_3 = 1$.

If $d_1 = d_2 = d_3 = 1$, we easily obtain that $a = b = c$, which contradicts the assumption that a, b and c are distinct. If $d_1 = -1$, then $a = c$, again, a contradiction.

(2) I have to apologize that this question is in fact wrong. Some of you discovered that Ramsey's theorem can be used to show that there exists a counterexample, that is, a bi-colored K_{10} may contain no monochromatic K_4. In particular, I'm very pleased with Chris's wonderful picture for this. See the course website for the scanned copy of his picture.

For those who pointed out the falsity, I gave full 3 points. Otherwise, I assigned 2.5 points, if you chose to work this question.

(3) First, one can show that $n^3 \equiv 0, 1$ or 8 mod 9, when $n \equiv 0, 1$ or 2 mod 3 respectively. Hence, modulo 9, we have
$$x^3 \equiv 0, 1 \text{ or } 8$$
$$2y^3 \equiv 0, 2 \text{ or } 7$$
$$4z^3 \equiv 0, 4 \text{ or } 5.$$
Therefore, $x^3+2y^3+4z^3 \equiv 0$ mod 9 can happen only when $x^3 \equiv y^3 \equiv z^3 \equiv 0$ mod 9, or $x \equiv y \equiv z \equiv 0$ mod 3. In this case, $x^3 + 2y^3 + 4z^3 \equiv 0$ mod 27, hence $x^3 + 2y^3 + 4z^3 = 9$ cannot be satisfied.

(4) Imagine that there are $6k$ *slots* on C, dividing C into $6k$ *blocks* of length 1. A slot is either *full* or *empty*, depending on whether or not a point is in it.

Assume that there exists a configuration of $3k$ points on C, such that no two points are antipodal. Then, an empty slot is always opposite to a full slot, and vice versa. To each block (of length 1), we assign $+1$ if the endpoints of the block are either both full or both empty. Otherwise, -1 is assigned. Call S the sum of these numbers for all $6k$ blocks. Then S is always equal to $-2k$; indeed, an arc of length 1 is a block which is assigned $+1$. A length 2 arc has two blocks, both of which got -1. If an arc is of length 3, it contains one "$+1$" block (in the middle) and two "-1" blocks.

Fix a full slot. Proceeding counterclockwise direction, perform the following operation to the next slot; if the next slot is empty, then exchange it with its opposite (full) slot (that is, fill it and empty the opposite). If it is full, don't do anything. And then move to the next one and do the same operation. Before we finish one cycle, the configuration of the points will be such that $3k$ consecutive slots are full and the rest $3k$ are empty. Then we stop performing the operation. Note that this operation doesn't change the property of C that an empty slot and a full slot are always opposite to each other.

The crux move is to keep track of the change of S at each operation. Let a_i be the slot under consideration. Note that a_{i-1} is always full because we began with a fixed full slot. If a_i is full, no operation is done, so S remains unchanged. Consider the two cases;

- a_i empty and a_{i+1} empty – In this case, the operation changes the configuration of a_{i-1}, a_i, a_{i+1} as

  $$\text{full, empty, empty} \Longrightarrow \text{full, full, empty.}$$

  And the change on the *opposite* side is

  $$\text{empty, full, full} \Longrightarrow \text{empty, empty, full.}$$

  Therefore, we see that S remains unchanged in this case.
- a_i empty and a_{i+1} full – similarly, the configuration of a_{i-1}, a_i, a_{i+1} is

  $$\text{full, empty, full} \Longrightarrow \text{full, full, full.}$$

  On the opposite side,

  $$\text{empty, full, empty} \Longrightarrow \text{empty, empty, empty.}$$

  Then, we see that S will increase by 8. Indeed, two -1 blocks around a_i become two $+1$ blocks, and the same thing happens opposite to a_i.

In conclusion, we see that S **modulo** 8 is unchanged during this process.

However, after we finish the process, all the full slots are on one side and the empty slots are on the other side, hence $S = 6k - 4$. But, obviously, $6k - 4 \not\equiv -2k$ modulo 8 for any integer k, therefore, a contradiction.