Homework 5

(1) Follow the method given in the lecture and find all the rational solutions to $x^2 + y^2 = 2$.

(2) Prove that there is no rational solutions to $x^2 + y^2 = 3$.

(3) (a) Use our *Main Theorem* to prove that the equation

$$C_d : x^2 - dy^2 = 1$$

admits a rational solution for any positive integer d. This equation is usually called *Pell equation*.

(b) Let $x = x_0$ and $y = y_0$ be *integral* solutions to the equation C_d. Show that one can obtain infinitely many integral solutions to C_d from (x_0, y_0).

Thus, the existence of one solution to a Pell equation immediately yields infinitely many solutions. However, sometimes it is difficult to find a *single* solution. For example, take the equation

$$x^2 - 4,729,494y^2 = 1$$

where 4,729,494 is a square-free integer, factored as

$$4,729,494 = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 29 \cdot 353.$$

This particular equation is related to the famous *Archimedes' Cattle Problem.*[1] The minimal integral solutions are known to be:

$$x = 109931986732829734979866232821433543901088049$$
$$y = 50549485234315033074477819735540408986340.$$

(4) In the ancient Greek book *Arithmetica* Diophantus says that the equation $15x^2 - 36 = y^2$ does not have a rational solution. Verify this using our Main Theorem.

(5) Let p be an odd prime. Prove that there exist $x, y \in \mathbf{Q}$ satisfying $p = x^2 + 26y^2$ if and only if $p \equiv 1$ or $3 \bmod 8$ and $p \equiv 1, 3, 4, 9, 10$ or $12 \bmod 13$.

[1] For the details to this interesting problem, see, for example, `http://www.mcs.drexel.edu/~crorres/Archimedes/Cattle/Statement.html`