Homework 4

(1) Prove Jensen's inequality; if f is convex on an interval I and if $x_1, \ldots, x_n$ are in I, then

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \frac{1}{n}\left(f(x_1) + \cdots + f(x_n)\right).$$

When does the equality hold?

(2) Show that one can derive AM-GM inequality from Jensen's inequality. Which convex function will you use?

(3) Prove that

$$x^x \geq \left(\frac{x+1}{2}\right)^{x+1}$$

for $x > 0$. Hint: The function $x \log x$ is convex on $(0, \infty)$.

(4) (Exercise 5.5.40) Let $a, b, c \geq 0$. Prove that

$$\sqrt{3(a+b+c)} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

When does the equality hold?

(5) (Exercise 6.1.25) Which are there more of among the natural numbers between 1 and 1,000,000; numbers that can be represented as a sum of a perfect square and a perfect cube, or numbers that cannot be?