Homework 3 for Putnam Problem Solving (110.225)

Due by October 21

(1) (Exercise 4.1.8) Show that every graph contains two vertices of equal degrees.

(2) (Exercise 4.2.30) Show that there do not exist any equilateral triangles in the plane whose vertices are lattice points.

(3) (Exercise 4.3.9) Prove that for any positive integers k less than m and n,

$$\sum_{j=0}^{k} \binom{n}{j} \binom{m}{k-j} = \binom{n+m}{k}.$$

(4) Let $n \geq 2$ be fixed and let $\zeta = \cos(2\pi/n) + i\sin(2\pi/n)$. (Here, i is the pure imaginary number with $i^2 = -1$.) Determine the (complex) numbers $a_0, a_1, \ldots, a_{n-1}$ to finish the partial fraction decomposition

$$\frac{1}{x^n - 1} = \sum_{k=0}^{n-1} \frac{a_k}{x - \zeta^k}.$$

(5) (This is a generalization of the problem we did in class.) Let $\zeta_1, \ldots, \zeta_n$ be all the n-th roots of unity. Evaluate

$$\prod_{i<j} (\zeta_i - \zeta_j)^2.$$

Note: For complex numbers, z^2 is *not* the same as $|z|^2$.