Homework 2 for Putnam Problem Solving (110.225)

Due by October 7

(1) Given three variables x, y, z, let

$$s_1 = x + y + z, \qquad s_2 = xy + yz + zx, \qquad s_3 = xyz.$$

Express the following as polynomials in s_1, s_2 and s_3.
  (a) $x^3 + y^3 + z^3$
  (b) $(x + y)(y + z)(z + x)$
  (c) $xy^4 + yz^4 + zx^4 + x^4y + y^4z + z^4x$

(2) Do Exercise 3.2.15 from the book. But turn in only part (d), which is the following; prove that

$$\mathrm{GCD}(a, b) = \min\{ax + by \mid x \text{ and } y \text{ integers such that } ax + by \text{ is positive}\}.$$

(3) Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients and let p be a prime. Assume that p is a factor of each of $a_{n-1}, \ldots, a_1, a_0$, but p is not a factor of a_n, and p^2 is not a factor of a_0. Show that $P(x)$ is irreducible over the rationals; that is $P(x)$ cannot be factored into two non-constant polynomials with rational coefficients.

(4) Let $P_1, \ldots, P_9$ be nine distinct lattice points (i.e. points with integral coordinates) in three dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of P_i's.

(5) Find positive integers n and $a_1, a_2, \ldots, a_n$ such that

$$a_1 + a_2 + \cdots + a_n = 2003$$

and the product $a_1 a_2 \cdots a_n$ is as large as possible.