Homework 1 for Putnam Problem Solving (110.225)

Due by September 23

Turn in the following 6 questions.

(1) Use mathematical induction to prove

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

for any positive integer n.

(2) Use mathematical induction to prove

$$f_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$

for any nonnegative integer n. Here, f_n is the n-th *Fibonacci number* defined by $f_0 = 0$, $f_1 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for any $n \geq 0$.

(3) The sequence $a_0, a_1, a_2, \ldots$ satisfies

$$a_{m+n} + a_{m-n} = \frac{1}{2}(a_{2m} + a_{2n})$$

for all nonnegative integers m and n with $m \geq n$. If $a_1 = 1$, determine a_{1995}. (Remark: The solution is sketched in the book! Recommendation — Try it without the book first; you will need to use some type of induction at some point. Do not read the book until you torture yourself long enough. When you write it up, try to write it in your own words, although you may basically repeat what the book does.)

(4) Let x be a real number such that $x+1/x$ is an integer. Prove that x^n+1/x^n is also an integer for any positive number n.

(5) An $n \times n$ matrix (square array) whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a *silver* matrix if, for each $i = 1, \ldots, n$, the i-th row and the i-th column together contain all the members of S. Show that silver matrices exist for infinitely many values of n.

(6) Let $x_1, x_2, x_3, \ldots$ be a sequence of nonzero real numbers satisfying

$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-1}}$$

for $n \geq 3$. Establish necessary and sufficient conditions on x_1 and x_2 for x_n to be an integer for infinitely many values of n.

Self-study: Read the rest of chapter 2, which is not covered in the lecture, and try as many problems as you can.

Disclaimer: All of the above questions are either from the text book, past competition problems or other sources, and none of them are the instructor's original work.