

TAKE-HOME FINAL EXAM

PUTNAM PROBLEM SOLVING

- (1) (3.4.22) Prove that it is impossible to choose three distinct integers a, b and c such that

$$a - b \mid b - c \quad b - c \mid c - a \quad c - a \mid a - b.$$

- (2) (4.1.14) A *complete* graph is one where there is an edge between every two vertices. The complete graph with r vertices is denoted by K_r . Show that if the edges of K_{10} are colored blue or red, then there must be 4 vertices, such that all the edges connecting them are the same color. In other words, “a bi-colored K_{10} must possess a monochromatic K_4 subgraph.”

- (3) Prove that there are no integers x, y and z such that

$$x^3 + 2y^3 + 4z^3 = 9.$$

Hint: For an integer n , what are the possible values for n^3 modulo 9?

- (4) Let k be a fixed natural number and let C be a circle whose circumference is of length $6k$. There are $3k$ distinct points on C . Of the $3k$ arcs formed by these points, assume that k arcs are of length 1, another k are of length 2 and the rest are of length 3. Prove that, at least, two of the $3k$ points are *antipodal*, that is, two of them are symmetric with respect to the center of C .

Thank you for your work in this class. Happy Holiday!